CONJECTURES FOR MOMENTS ASSOCIATED WITH CUBIC TWISTS OF ELLIPTIC CURVES

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Abstract. We extend the heuristic introduced by Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+05] in order to formulate conjectures for the (k, ℓ)-moments of L-functions of elliptic curves twisted by cubic characters. We also apply the work of Keating and Snaith [KS00] on the (k, ℓ)-moments of characteristic polynomials of unitary matrices to extend our conjecture to $k, \ell \in \mathbb{C}$ such that $\mathrm{Re}(k), \mathrm{Re}(\ell)$, and $\mathrm{Re}(k + \ell) > -1$. Our conjectures are then numerically tested for two families.

1. Introduction

We present in this paper conjectures for the general moments of L-functions of elliptic curves twisted by cubic characters, and test them numerically. More precisely, let E be a fixed elliptic curve over $\mathbb{Q}$ with conductor N_E and L-function $L(s, E)$ as defined by (13). This is the normalized L-function such that the functional equation relates $L(s, E)$ to $L(1-s, E)$ and the central critical value is $L(1/2, E)$. Let $\mathcal{F}_E$ is the set of primitive cubic characters defined over $\mathbb{Q}$ with conductor co-prime to $3N_E$. We also denote by $\mathcal{F}_E(D)$ the subset of $\mathcal{F}_E$ consisting of characters with conductor less or equal to D. For any $\chi \in \mathcal{F}_E$, the twisted L-function is defined by

$$L(s, E, \chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}, \quad \text{where} \quad L(s, E) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

By the Hasse bound, both series converge for $\mathrm{Re}(s) > 1$, and have analytic continuation to the whole complex plane by the work of Wiles [Wil95].

We define the general (k, ℓ)-moment as

$$\left\langle L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell \right\rangle_D := \frac{1}{\#\mathcal{F}_E(D)} \sum_{\chi \in \mathcal{F}_E(D)} L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell.$$

There are few results in the literature for moments of cubic twists of L-functions. For cubic twists of L-functions attached to automorphic representations of $\mathrm{GL}(2, \mathbb{A}_K)$, where $K = \mathbb{Q}(\sqrt{-3})$, a weighted non-sieved first moment was computed was computed by Brubacker, Friedberg, and Hoffstein in [BFH05]. This is the only result for cubic twist of L-functions of elliptic curves. For Dirichlet cubic twists, the first moment (the $(1, 0)$-moment) for cubic characters over $\mathbb{Q}$ was computed by Baier and Young [BY10] (the non-Kummer case), and by Luo for a subset of the cubic characters over $\mathbb{Q}(\sqrt{-3})$ [Luo04] (the Kummer case). For

2010 *Mathematics Subject Classification.* Primary 11M41; Secondary 11M50, 14G10, 11M26.

Key words and phrases. L-functions; moments; cubic twists; random matrix theory.

This work was supported by the Natural Sciences and Engineering Research Council of Canada - NSERC [Discovery Grant 155635-2019 to CD, 355412-2013 to ML] and the Fonds de recherche du Québec - Nature et technologies - FRQNT [Projet de recherche en équipe 256442 to CD and ML].

the case of function fields, the first moment was computed by David, Florea, and Lalín in both the Kummer and non-Kummer case [DFL19]. The scarcity of results for moments of L-functions twisted by cubic characters, compared to the abundance of results for moments of L-functions twisted by quadratic characters, can be explained by the fact that the sign of the functional equation of cubic twists involves cubic Gauss sums which are chaotic objects.

We conjecture that as $D \to \infty$,

$$\left\langle L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell \right\rangle_D \sim \frac{g_{k,\ell}\, a_{k,\ell}}{(k\ell)!} \log^{k\ell} D,$$

where $g_{k,\ell}$ is given by (1) and depends only on the symmetry type of the family (coming from random matrix theory), and the arithmetic factor $a_{k,\ell}$ is given by (3) and depends on the arithmetic of the family. In fact, we make a conjecture for all the powers of $\log D$ and we compute explicitly the first two terms, see Conjecture 1.1.

The division by $(k\ell)!$ is a standard normalization which makes the random matrix factor $g_{k,\ell}$ an integer. Families of cubic twists are unitary families, and for unitary families, previous literature considers mostly the (k,k)-moments, which in our case would be

$$\frac{1}{\#\mathcal{F}_E(D)} \sum_{\chi \in \mathcal{F}_E(D)} |L(1/2, E, \chi)|^{2k}.$$

In the case of cubic twists, all moments are real, even if the special values $L(1/2, E, \chi)$ are complex numbers, since $L(1/2, E, \overline{\chi}) = \overline{L(1/2, E, \chi)}$, and it is natural to consider all (k,ℓ)-moments.

Independently of our conjecture, the random matrix factor $g_{k,\ell}$ can be deduced from the work of Keating and Snaith [KS00], as we explain in Section 2. In both cases, this gives

$$g_{k,\ell} := (k\ell)! \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!}. \tag{1}$$

Notice that

$$g_{k,k} = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}, \tag{2}$$

which is the usual g_k associated with the (k,k) -moments of unitary matrices, as computed in [KS00].

We now state our main result, a conjecture for the (k,ℓ)-moments of cubic twists of elliptic curves L-functions, with an explicit formula for the first 2 terms.

Conjecture 1.1. *As $D \to \infty$,*

$$\frac{1}{\#\mathcal{F}_E(D)} \sum_{\chi \in \mathcal{F}_E(D)} L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell = P_{k\ell}(\log D) + O(D^{-\delta}),$$

for some $\delta > 0$, and where $P_{k\ell}(x) = c_{k\ell} x^{k\ell} + c_{k\ell-1} x^{k\ell-1} + \cdots + c_0$ is a polynomial of degree $k\ell$ with

$$c_{k\ell} = \frac{g_{k,\ell}\, a_{k,\ell}}{(k\ell)!} \quad \text{and} \quad a_{k,\ell} = 2^{k\ell} A_{E,k,\ell}(0,0), \tag{3}$$

where

$$A_{E,k,\ell}(z_1,z_2)=\prod_p A_{E,k,\ell}(z_1,z_2;p),$$

and the Euler factors are defined by (35)*. Furthermore, we also compute that*

$$\begin{aligned} c_{k\ell-1} =& g_{k,\ell}2^{k\ell-1} \\ &\times\left(\left(\gamma(k+\ell-2)+\log\left(\frac{N_E}{4\pi^2}\right)-2\right)A_{E,k,\ell}(0,0)+\left(\frac{1}{k}\frac{\partial A_{E,k,\ell}}{\partial z_1}(0,0)-\frac{1}{\ell}\frac{\partial A_{E,k,\ell}}{\partial z_2}(0,0)\right)\right). \end{aligned}$$

We deduce our conjectures from a general framework based on the paper [CFK$^+$05] often called "the recipe" (see also [AK14, RW15] for the function field setting). It is based on computing the *shifted moments*, and using the combinatorics given by the shifts to get the conjecture for the moments. There is a strong analogy with shifted moments of unitary random matrices as explained in Section 2. The recipe leads to the conjecture (16) stated at the beginning of Section 4. In Section 5, we proceed to compute formulas for the first 2 coefficients from (16), generalizing the work of [CFK$^+$05] to the case $k\neq\ell$ to compute the first coefficient $c_{k\ell}$, and extending our formulas to compute the second coefficient $c_{k\ell-1}$. We remark that Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK$^+$08] computed the first 10 coefficients for the case of the zeta function, which also corresponds to the unitary case, while Rubinstein and Yamagishi [RY15] obtained the first 170 coefficients numerically in this case. In fact, our formula for $c_{k\ell-1}$ is analogous to formula (2.71) from [CFK$^+$08], but in our case, we have $k\neq\ell$ and the arithmetic factor is considerably different. Goulden, Huynh, Rishikesh, and Rubinstein [GHRR13] computed several coefficients for quadratic Dirichlet L-functions and L-functions associated to quadratic twists of an elliptic curve over $\mathbb{Q}$, corresponding to symplectic and orthogonal families.

Our work differs from the works cited above because we consider cubic characters and our situation allows for the possibility of $k\neq\ell$, leading to considerations of entirely new moments. The case where either k or ℓ are 0 is much simpler, and is presented in Section 6.

The random matrix theory factor is naturally extended to all $k,\ell\in\mathbb{C}$ such that $\mathrm{Re}(k),\mathrm{Re}(\ell)$, and $\mathrm{Re}(k+\ell)>-1$, since all (k,ℓ)-moments of characteristic polynomials of unitary random matrices can be computed in this region from the work of Keating and Snaith [KS00], as explained in Section 2. Our formulas for the second coefficient $c_{k\ell-1}$ are written in such a way that they can also be extended for non-integer values of k and ℓ, and we speculate that Conjecture 1.1 could also be extended to $k,\ell\in\mathbb{C}$ such that $\mathrm{Re}(k),\mathrm{Re}(\ell)$, and $\mathrm{Re}(k+\ell)>-1$ in Section 7. In that case, $P_{k\ell}(x)$ is the power series $c_{k\ell}x^{k\ell}+c_{k\ell-1}x^{k\ell-1}+\cdots$. We remark that the extension to non-integral powers was already considered in [CFK$^+$08] for the zeta function. Our case differs because having $k\neq\ell$ allows for the possibility that one of the parameters is negative, such as in the case of $k=1/2,\ell=-1/2$.

We present in Section 8 numerical tests for several values of (k,ℓ), including real non-integers, and cases involving a negative values for one or two parameters, and complex parameters, for two elliptic curves. As explained in Section 8, the amount of data that can be obtained for cubic twists is unfortunately limited compared to the case of quadratic twists, but the numerical tests still show a good fit with the conjectures.

Acknowledgments: The authors would like to thank Hershy Kisilevsky for many useful discussions, Andrew Granville for suggesting the formula of Lemma 2.2 to show that $g_{k,\ell}$ is an integer, Jon Keating and Michael Rubinstein for bringing our attention to the status of the computations of coefficients for zeta functions and quadratic twists, and Pablo Bianucci for helpful advice in some of the numerical computations. The authors are very grateful to the Centre de Recherches Mathématiques (CRM) for the access to their cluster and to Jack Fearnley for providing his L-value data.

2. Moments of characteristic polynomials of random matrices

In their influential work, Keating and Snaith concentrated on the (k,k)-moments of unitary matrices [KS00]. For a unitary matrix U let $Z_U(s)$ be the characteristic polynomial

$$Z_U(s) = \det\left(I - Us\right).$$

For any positive integer k, they prove that the moments over the set of all $N \times N$ unitary matrices (denoted $U(N)$) with respect to the Haar measure satisfy

$$\left\langle |Z_U|^{2k} \right\rangle_{U(N)} := \int_{U(N)} \left|Z_U(e^{-i\theta})\right|^{2k} \, d\mathrm{Haar}(U) \sim \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} N^{k^2} = \frac{g_{k,k}}{k^2!} N^{k^2}$$

as $N \to \infty$ [KS00, Equations (15, 16), page 60]. Notice that the answer is independent of the choice of θ. We now show how to deduce deduce from their work that for the (k,ℓ)-moments,

$$\left\langle Z_U^k \overline{Z}_U^\ell \right\rangle_{U(N)} \sim \frac{G(k+1)G(\ell+1)}{G(k+\ell+1)} N^{k\ell} \tag{4}$$

as $N \to \infty$ (for any $k, \ell \in \mathbb{C}$ such that $\mathrm{Re}(k), \mathrm{Re}(\ell), \mathrm{Re}(k+\ell) > -1$).

In the above, G denotes the Barnes G-function, which is given by the following Weierstrass product

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right) \right\},$$

where γ is the Euler–Mascheroni constant.

It satisfies the following crucial properties

$$G(z+1) = \Gamma(z)G(z), \quad G(0) = 0, \quad G(1) = 1, \quad G(n) = \prod_{j=0}^{n-2} j! \tag{5}$$

Writing

$$\left\langle Z_U^k \overline{Z_U}^\ell \right\rangle_{U(N)} = \left\langle |Z_U|^{k+\ell} e^{i(k-\ell)\mathrm{Im}\log Z_U} \right\rangle_{U(N)}, \tag{6}$$

and using [KS00, Equation (71), page 71], we get

$$\left\langle Z_U^k \overline{Z_U}^\ell \right\rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(k+\ell+j)}{\Gamma(j+k)\Gamma(j+\ell)}. \tag{7}$$

Proposition 2.1. *As $N \to \infty$,*

$$\prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(k+\ell+j)}{\Gamma(j+k)\Gamma(j+\ell)} \sim \frac{G\left(k+1\right) G\left(\ell+1\right)}{G(k+\ell+1)} N^{k\ell}.$$

Proof. First notice that

$$\prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(k+\ell+j)}{\Gamma(j+k)\Gamma(j+\ell)} = \frac{G\,(k+1)\,G\,(\ell+1)}{G(k+\ell+1)} \frac{G(N+1)G(k+\ell+N+1)}{G(k+N+1)G(\ell+N+1)}.$$

We use Barnes' formula

$$\log G(z+1) = \frac{z^2}{2}\log z - \frac{3z^2}{4} + \frac{z}{2}\log(2\pi) - \frac{1}{12}\log z + \left(\frac{1}{12} - \log A\right) + O\left(\frac{1}{z^2}\right) \tag{8}$$

to estimate the quotient of the Barnes G-functions evaluated at N. Thus we consider

$$\lim_{N\to\infty} \log G(N+1) + \log G(k+\ell+N+1) - \log G(k+N+1) - \log G(\ell+N+1).$$

All the terms approach zero when $N \to \infty$, except for those coming from $-\frac{3z^2}{4}$ and $\frac{z^2}{2}\log z$.

The terms of the form $-\frac{3z^2}{4}$ give

$$-\frac{3}{4}\left(N^2 + (k+\ell+N)^2 - (k+N)^2 - (\ell+N)^2\right) = -\frac{3k\ell}{2},$$

and the terms of the form $\frac{z^2}{2}\log z$ give

$$\frac{1}{2}\left(N^2\log N + (k+\ell+N)^2\log(k+\ell+N) - (k+N)^2\log(k+N) - (\ell+N)^2\log(\ell+N)\right)$$
$$\sim -\frac{k\ell}{2} + 2k\ell + k\ell\log(k+\ell+N).$$

Combining everything, we get

$$\log\left(\frac{G(N+1)G(k+\ell+N+1)}{G(k+N+1)G(\ell+N+1)}\right) \sim k\ell\log(k+\ell+N)$$

and we deduce the result by taking the exponential and observing that $(N+k+\ell)^{k\ell} \sim N^{k\ell}$ as $N \to \infty$. $\square$

Using Proposition 2.1 in (7), we get (4), and using (5) it follows that for $k, \ell \in \mathbb{Z}$

$$\left\langle Z^k \overline{Z}^\ell \right\rangle_{U(N)} \sim \frac{g_{k,\ell}}{(k\ell)!} N^{k\ell}.$$

For the sake of completeness, we include the proof that $g_{k,\ell}$ is an integer, which follows by a counting argument (see the work of Connery and Farmer [CF00] for the proof in the case $k = \ell$).

Lemma 2.2. *For any positive integer q, let $k \equiv k_q \bmod q$ and $\ell \equiv \ell_q \bmod q$ with $1 \le k_q, \ell_q \le q$. Then,*

$$g_{k,\ell} = \prod_{q=p^n} p^{e(q)}, \tag{9}$$

where the product runs over prime powers, and

$$e(q) = \begin{cases} \left[\frac{k_q\ell_q}{q}\right] & k_q + \ell_q \le q, \\ \left[\frac{(q-k_q)(q-\ell_q)}{q}\right] & k_q + \ell_q > q. \end{cases} \tag{10}$$

It follows that for $k, \ell \in \mathbb{Z}_{>0}$, $g_{k,\ell} \in \mathbb{Z}$.

Proof. We write

$$g_{k,\ell} = \frac{k\ell \cdot (k\ell-1)\dots 1 \cdot 1^{k-1}2^{k-2}\dots(k-1)\cdot 1 \cdot 1^{\ell-1}2^{\ell-2}\dots(\ell-1)}{1^{k+\ell-1}2^{k+\ell-2}\dots(k+\ell-1)}.$$

For any positive integer q, let $n(q)$ and $d(q)$ be the number of integers (counted with multiplicities) on the numerator, respectively the denominator, which are multiples of q. Let $e(q) = n(q) - d(q)$. Then, $g_{k,\ell}$ is given by (9), and we have to show the formulas for $e(p)$ as stated in the lemma. We have that

$$\begin{aligned} n(q) &= \left[\frac{k\ell}{q}\right] + \sum_{i=1}^{[(k-1)/q]} (k-qi) + \sum_{i=1}^{[(\ell-1)/q]} (\ell-qi) \\ d(q) &= \sum_{i=1}^{[(k+\ell-1)/q]} (k+\ell-qi) \end{aligned}$$

We write $k = qa_q + k_q$ and $\ell = qb_q + \ell_q$ where $1 \le k_q, \ell_q \le q$. Then,

$$\begin{aligned} [(k-1)/q] &= a_q \\ [(\ell-1)/q] &= b_q \\ [(k+\ell-1)/q] &= \begin{cases} a_q + b_q & k_q + \ell_q \le q \\ a_q + b_q + 1 & k_q + \ell_q > q. \end{cases} \end{aligned}$$

(11) $$[k\ell/q] = qa_qb_q + a_q\ell_q + k_qb_q + [k_q\ell_q/q]$$

We first suppose that $k_q + \ell_q \le q$. In that case,

$$\begin{aligned} e(q) &= \left[\frac{k\ell}{q}\right] + ka_q + \ell b_q - (k+\ell)(a_q+b_q) - \frac{q}{2}\left(a_q(a_q+1) + b_q(b_q+1) - (a_q+b_q)(a_q+b_q+1)\right) \\ &= \left[\frac{k\ell}{q}\right] - (a_qq+k_q)b_q - (b_qq+\ell_q)a_q + qa_qb_q \\ &= \left[\frac{k_q\ell_q}{q}\right], \end{aligned}$$

where the last line follows by using (11).

If $k_q + \ell_q > q$, we have

$$\begin{aligned} e(q) &= \left[\frac{k\ell}{q}\right] + ka_q + \ell b_q - (k+\ell)(a_q+b_q+1) - \frac{q}{2}\left(a_q(a_q+1) + b_q(b_q+1)\right. \\ &\quad \left. -(a_q+b_q+1)(a_q+b_q+2)\right) \\ &= \left[\frac{k\ell}{q}\right] - (a_qq+k_q)b_q - (b_qq+\ell_q)a_q - (k+\ell) + qa_qb_q + q(a_q+b_q+1) \\ &= \left[\frac{k_q\ell_q}{q}\right] + qa_q + qb_q + q - k - \ell \\ &= \left[\frac{(q-k_q)(q-\ell_q)}{q}\right]. \end{aligned}$$

Since $e(q) \geq 0$ for all q, it follows that $g_{k,\ell}$ is an integer.

□

An alternative proof that $g_{k,\ell} \in \mathbb{Z}$ for $k, \ell \in \mathbb{Z}_{>0}$ can be obtained by observing that $g_{k,\ell}$ counts the number of rectangular standard Young tableaux of $k \times \ell$, which can be deduced from the Hook length formula.

Finally, we state the result for the *shifted* (k, ℓ)-moments of unitary random matrices which is the base for the recipe, where analogous computations are performed on the L-functions. This is the extension of [CFK+05, Theorem 1.5.2], which is stated for the (k, k)-moments. Since U is unitary, we have the functional equation

$$Z_U(s) = \varepsilon(U) s^N Z_{U^*}(s^{-1}),$$

where U^* is the transpose conjugate matrix, and the sign of the functional equation is $\varepsilon(U) = (-1)^N \det U$. Let

$$\mathcal{Z}_U(s) = \varepsilon(U)^{-1/2} s^{-N/2} Z_U(s).$$

The following theorem is proven in [CFK+03]. The integrand contains terms of the form $(1 - e^{-z_h + z_{k+m}})^{-1}$, which have simple poles at $z_{k+m} = z_h$. For the L-functions, the simple poles will come from the arithmetic factor, which contains the terms $\zeta(1 + z_h - z_{k+m})$ after extracting the poles. See (28). In both cases (the theorem below for random matrices and the conjecture for L-functions), the formulas are derived from Lemma 4.3 (Lemma 2.5.3 from [CFK+05]) and they involve the Vandermonde determinant

$$\Delta(z_1, \ldots, z_m) = \begin{vmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_m \\ \vdots & \ddots & \vdots \\ z_1^{m-1} & \cdots & z_m^{m-1} \end{vmatrix} = \prod_{1 \leq i < j \leq m} (z_j - z_i). \tag{12}$$

Theorem 2.3. *Let* $\alpha = (\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{k+\ell})$ *and*

$$G(z_1, \ldots, z_{k+\ell}) = \prod_{h=1}^{k} \prod_{m=1}^{\ell} \left(1 - e^{-z_h + z_{k+m}}\right)^{-1}.$$

Then,

$$\left\langle \mathcal{Z}_U(e^{-\alpha_1}) \cdots \mathcal{Z}_U(e^{-\alpha_k}) \mathcal{Z}_{U^*}(e^{\alpha_{k+1}}) \cdots \mathcal{Z}_{U^*}(e^{\alpha_{k+\ell}}) \right\rangle_{U(N)} = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!} \frac{1}{(2\pi i)^{k+\ell}} \times$$
$$\times \oint \cdots \oint \frac{G(z_1, \ldots, z_{k+\ell}) \Delta(z_1, \ldots, z_{k+\ell})^2}{\prod_{h=1}^{k+\ell} \prod_{m=1}^{k+\ell} (z_h - \alpha_m)} e^{\frac{N}{2}\left(\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}\right)} \, dz_1 \ldots dz_{k+\ell},$$

where the integration takes place over small circles around $\alpha_1, \ldots, \alpha_{k+\ell}$.

3. Estimates for the number of characters and moments of conductors

Let E be an elliptic curve with conductor N_E and let $L(s, E)$ be its L-function

$$L(s, E) = \prod_{p \nmid N_E} \left(1 - a_p p^{-s} + p^{-2s}\right)^{-1} \prod_{p \mid N_E} \left(1 - a_p p^{-s}\right)^{-1}, \tag{13}$$

where we take

$$a_p = \begin{cases} \frac{p+1-\#E(\mathbb{F}_p)}{\sqrt{p}} & p \nmid N_E, \\ \\ \pm\frac{1}{\sqrt{p}}, 0 & p \mid N_E. \end{cases} \tag{14}$$

Notice that this is the normalization such that the central critical value is $L(1/2, E)$. From the Hasse bound, the L-function converges absolutely for $\mathrm{Re}(s) > 1$, and from [Wil95, TW95], it has analytic continuation for all $s \in \mathbb{C}$ and it satisfies the functional equation

$$L(s,E) = \omega_E \frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)} \left(\frac{4\pi^2}{N_E}\right)^{s-\frac{1}{2}} L(1-s,E),$$

where the sign of the functional equation is $\omega_E = \pm 1$.

For any fixed integer N, let $\mathcal{F}_N$ be the set of primitive cubic characters with conductor co-prime to $3N$, and let $\mathcal{F}_N(D)$ be the subset of those characters with conductor less or equal to D. This following lemma can be found in [Coh54] for the family of all cubic characters, but in our case the constant is slightly different as we are excluding characters of conductor not co-prime to $3N$.

Lemma 3.1.

$$\#\mathcal{F}_N(D) \sim c_3(N) D,$$

where

$$c_3(N) = \frac{\sqrt{3}}{2\pi} \prod_{p \equiv 1 \pmod 3} \frac{(p+2)(p-1)}{p(p+1)} \prod_{\substack{p \equiv 1 \pmod 3 \\ p \mid N}} \left(1 + \frac{2}{p}\right)^{-1}.$$

Proof. Let $\zeta_K(s)$ is the Dedekind zeta function of $K = \mathbb{Q}(\sqrt{-3})$, which is

$$\zeta_K(s) = \left(1 - \frac{1}{3^s}\right)^{-1} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

It has a simple pole at $s = 1$ with residue $\frac{\pi}{3\sqrt{3}}$. We write the generating series for $\#\mathcal{F}_N(D)$ as

$$G_N(s) \;=\; \prod_{\substack{p \equiv 1 \pmod 3 \\ p \nmid 3N}} \left(1 + \frac{2}{p^s}\right) = \zeta_K(s) F_N(s),$$

where

$$F_N(s) = \left(1 - \frac{1}{3^s}\right) \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{2}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^2 \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^{2s}}\right) \prod_{\substack{p \equiv 1 \pmod 3 \\ p \mid N}} \left(1 + \frac{2}{p^s}\right)^{-1}$$

is analytic for $\mathrm{Re}(s) > 1/2$. Using the Tauberian theorem, this shows the result. Using Perron's formula and bounding $\zeta_K(s)$ in the critical critical strip gives a power saving error term, but we do not need this in our heuristic argument. ∎

Corollary 3.2. *As $D \to \infty$,*

$$\frac{\#\mathcal{F}_{mN}(D)}{\#\mathcal{F}_N(D)} \sim \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 3)\\ p|m, p\nmid N}} \left(1+\frac{2}{p}\right)^{-1}.$$

For any function $f(d)$ defined on positive integers, let

$$\langle f(d)\rangle_D = \frac{1}{\#\mathcal{F}(D)} \sum_{\chi\in\mathcal{F}(D)} f(cond(\chi)).$$

Then for h a real number and n a nonnegative integer,

$$\left\langle \log^h d\right\rangle_D \sim \sum_{j=0}^{n}(-1)^j h(h-1)\cdots(h-j+1)\log^{h-j} D + O\left(\log^{h-n-1} D\right). \tag{15}$$

Moreover, when h is a nonnegative integer and $n=h$, we have

$$\left\langle \log^h d\right\rangle_D \sim h! \sum_{j=0}^{h} \frac{(-1)^j}{(h-j)!} \log^{h-j} D.$$

Proof. We have that

$$\frac{\#\mathcal{F}_{mN}(D)}{\#\mathcal{F}_N(D)} \sim \frac{c_3(mN)}{c_3(N)},$$

and the first result follows.

For the average of $\log^h d$ results, using partial summation, we have that

$$\begin{aligned}\sum_{\chi\in\mathcal{F}_N(D)} \log^h d &\sim c_3(N) D\log^h D - h\int_1^D c_3(N)\log^{h-1} t\, dt \\ &\sim c_3(N)D\log^h D - c_3(N)hD\sum_{j=1}^{n}(-1)^{j-1}(h-1)\cdots(h-j+1)\log^{h-j} D \\ &\quad + O\left(D\log^{h-n-1} D\right)\end{aligned}$$

and the result follows. □

4. Shifted moments

We explain in this section how the recipe of [CFK⁺05] leads to the conjecture

$$\left\langle L(1/2,E,\chi)^k L(1/2,E,\overline{\chi})^\ell\right\rangle_D \sim \left\langle \Upsilon_{k,\ell}(2\log d)\right\rangle_D, \tag{16}$$

where

$$\Upsilon_{k,\ell}(x) = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!(2\pi i)^{k+\ell}} \oint\cdots\oint \frac{G(z_1,\dots,z_{k+\ell})\Delta(z_1,\dots,z_{k+\ell})^2}{(z_1\cdots z_{k+\ell})^{k+\ell}} e^{\frac{x}{2}\left(\sum_{h=1}^k z_h - \sum_{m=1}^{\ell} z_{k+m}\right)} dz_1\dots dz_{k+\ell},$$

and $G(z_1,\dots,z_{k+\ell})$ is defined by (27).

This is obtained by considering shifted moments and solving the combinatorics of the shifts as in the corresponding random matrix theorem (Theorem 2.3). Taking the limit when the shifts go to 0 leads to (16).

We assume that $k \ge \ell \ge 1$. The case where $\ell = 0$ is much simpler, and it is treated in Section 6.

Let χ be a character of conductor d. The functional equation for the twist of $L(s, E)$ by χ is given by

$$L(s, E, \chi) = \varepsilon(E, \chi)\frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma\left(s + \frac{1}{2}\right)}\left(\frac{4\pi^2}{N_E}\right)^{s-\frac{1}{2}} d^{1-2s}L(1 - s, E, \overline{\chi}), \tag{17}$$

where N_E is the conductor of E. The sign of the functional equation is

$$\varepsilon(E, \chi) = \frac{\omega_E\chi(N_E)\tau(\chi)^2}{d}, \tag{18}$$

where $\tau(\chi)$ is the Gauss sum of the character. We take

$$Z(s, E, \chi) = \mathcal{X}(s, E, d)^{-1/2}L(s, E, \chi),$$

where

$$\mathcal{X}(s, E, d) = \frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma\left(s + \frac{1}{2}\right)}\left(\frac{4\pi^2}{N_E}\right)^{s-\frac{1}{2}} d^{1-2s} =: X(s, E)\ d^{1-2s}, \tag{19}$$

and the functional equation becomes

$$Z(s, E, \chi) = \varepsilon(E, \chi)Z(1 - s, E, \overline{\chi}).$$

We remark the following result, which will be useful later.

Lemma 4.1.

$$\frac{\partial}{\partial s}(X(s, E)^{-1/2})\Big|_{s=\frac{1}{2}} = -\gamma + \frac{1}{2}\log\left(\frac{N_E}{4\pi^2}\right),$$

where γ is the Euler–Mascheroni constant.

Proof. First notice that

$$\begin{aligned} X'(s, E) =& \frac{-\Gamma'\left(\frac{3}{2} - s\right)\Gamma\left(s + \frac{1}{2}\right) - \Gamma\left(\frac{3}{2} - s\right)\Gamma'\left(s + \frac{1}{2}\right)}{\Gamma\left(s + \frac{1}{2}\right)^2}\left(\frac{4\pi^2}{N_E}\right)^{s-\frac{1}{2}} \\ &+ \frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma\left(s + \frac{1}{2}\right)}\left(\frac{4\pi^2}{N_E}\right)^{s-\frac{1}{2}}\log\left(\frac{4\pi^2}{N_E}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial s}(X(s, E)^{-1/2})\Big|_{s=\frac{1}{2}} &= -\frac{X'(1/2, E)}{2X(1/2, E)^{3/2}} = -\frac{1}{2}X'(1/2, E) \\ &= -\gamma + \frac{1}{2}\log\left(\frac{N_E}{4\pi^2}\right), \end{aligned}$$

where we have used the fact that $\Gamma'(1) = -\gamma$.

□

The approximate functional equation gives

$$Z(s, E, \chi) = \mathcal{X}(s, E, d)^{-1/2}\sum\frac{a_n\chi(n)}{n^s} + \varepsilon(E, \chi)\mathcal{X}(1 - s, E, d)^{-1/2}\sum\frac{a_n\overline{\chi}(n)}{n^{1-s}} \tag{20}$$

and

(21) $Z(s,E,\overline{\chi}) = \mathcal{X}(s,E,d)^{-1/2}\sum \frac{a_n\overline{\chi}(n)}{n^s} + \overline{\varepsilon(E,\chi)}\mathcal{X}(1-s,E,d)^{-1/2}\sum \frac{a_n\chi(n)}{n^{1-s}},$

where we have used the fact that (18) implies that

$$\varepsilon(E,\overline{\chi}) = \overline{\varepsilon(E,\chi)}.$$

In the approximate functional equations above, we neglected the smoothing and the bounds on the sum. For the exact formula we refer to Exercise 2 in page 99 in [IK04].

Notice that $\mathcal{X}(1/2,E,d) = 1$ implies that

$$\left\langle L(1/2,E,\chi)^k L(1/2,E,\overline{\chi})^\ell\right\rangle_D = \left\langle Z(1/2,E,\chi)^k Z(1/2,E,\overline{\chi})^\ell\right\rangle_D, \tag{22}$$

and therefore we can work with the average of Z instead of L.

Let $\alpha = (\alpha_1,\ldots,\alpha_{k+\ell})$. We consider the shifted moment

$$\left\langle Z(1/2,E,\chi)^k Z(1/2,E,\overline{\chi})^\ell\right\rangle_D^{(\alpha)} :=$$
$$\langle Z(1/2+\alpha_1,E,\chi)\cdots Z(1/2+\alpha_k,E,\chi)Z(1/2-\alpha_{k+1},E,\overline{\chi})\cdots Z(1/2-\alpha_{k+\ell},E,\overline{\chi})\rangle_D.$$

Notice that the α_j's with the conjugate characters are negative in accordance to Theorem 2.3. This pairs principal terms with principal terms and dual terms with dual terms and simplifies the computation.

We apply the approximate functional equations (20) and (21) to evaluate the products of L-functions. We obtain a sum of $2^{k+\ell}$ terms, but we keep only the terms where there is no oscillation on the product of signs of the functional equation, as we conjecture that the terms with oscillation do not contribute to the moment. This means that the product of the $\varepsilon(E,\chi)$ and $\overline{\varepsilon(E,\chi)}$ is 1, i.e. we keep only the terms where there is the same number of each. To make that precise, let $0 \le m \le \ell$ be the number of times that we are choosing the second factor of the approximate functional equation (21). Then, we must also choose the second factor of the approximate functional equation (20) m times, and this gives the average of terms of the following form:

$$\prod_{j=1}^k \mathcal{X}(1/2+e_j\alpha_j,E,d)^{-1/2}\prod_{j=1}^\ell \mathcal{X}(1/2+e_{j+k}\alpha_{j+k},E,d)^{-1/2}\sum_{n_1,\ldots,n_{k+\ell}} \frac{a_{n_1}\cdots a_{n_{k+\ell}}\,\chi^{e_1}(n_1)\cdots\chi^{e_{k+\ell}}(n_{k+\ell})}{n_1^{1/2+e_1\alpha_1}\cdots n_{k+\ell}^{1/2+e_{k+\ell}\alpha_{k+\ell}}},$$

where $(e_1,\ldots,e_{k+\ell}) \in \{\pm 1\}^{k+\ell}$ with the property that exactly ℓ of the e_i are equal to -1, which is independent of m. For example, for $m = 0$, we have

$$M(1/2;\alpha_1,\ldots,\alpha_{k+\ell}) = \prod_{j=1}^k \mathcal{X}(1/2+\alpha_j,E,d)^{-1/2}\prod_{j=1}^\ell \mathcal{X}(1/2-\alpha_{j+k},E,d)^{-1/2}$$
$$\times \sum_{n_1,\ldots,n_{k+\ell}} \frac{a_{n_1}\cdots a_{n_{k+\ell}}\,\chi(n_1)\cdots\chi(n_k)\overline{\chi}(n_{k+1})\cdots\overline{\chi}(n_{k+\ell})}{n_1^{1/2+\alpha_1}\cdots n_k^{1/2+\alpha_k}n_{k+1}^{1/2-\alpha_{k+1}}\cdots n_{k+\ell}^{1/2-\alpha_{k+\ell}}}. \tag{23}$$

Summing over all m, and counting the number of ways that we can choose the second factor of (21) m times, and the second factor of (20) m times, there is a total of

$$\sum_{m=0}^{\ell}\binom{\ell}{m}\binom{k}{m} = \binom{k+\ell}{\ell}$$

summands, which naturally recovers the number of $(k+\ell)$-uples $(e_1, \dots, e_{k+\ell})$.

We write the summands by permuting the shifts in $M(1/2; \alpha_1, \dots, \alpha_{k+\ell})$. Notice that $M(1/2; \alpha_1, \dots, \alpha_{k+\ell})$ is symmetric in $\alpha_1, \dots, \alpha_k$ and in $\alpha_{k+1}, \dots, \alpha_{k+\ell}$. Then, to get all the summands, it suffices to consider all the permutations in the cosets of $\mathrm{Sym}(k) \times \mathrm{Sym}(\ell)$ as a subgroup of $\mathrm{Sym}(k+\ell)$. We then conjecture that

$$\left\langle Z(1/2, E, \chi)^k Z(1/2, E, \overline{\chi})^\ell \right\rangle_D^{(\alpha)} = \left\langle \sum_{\sigma \in \Xi_{k,\ell}} M(1/2; \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k+\ell)}) \right\rangle_D, \tag{24}$$

where

$$\Xi_{k,\ell} = \mathrm{Sym}(k+\ell)/\mathrm{Sym}(k) \times \mathrm{Sym}(\ell).$$

According to the recipe of [CFK$^+$05], we now replace

$$\chi(n_1) \dots \chi(n_k) \overline{\chi}(n_{k+1}) \dots \overline{\chi}(n_{k+\ell})$$

in (23) by the average over the family using the following result.

Lemma 4.2. *As $D \to \infty$,*

$$\begin{aligned} \langle \chi(n_1) \cdots \chi(n_k) \overline{\chi}(n_{k+1}) \cdots \overline{\chi}(n_{k+\ell}) \rangle_D =& \langle \chi(n_1) \cdots \chi(n_k) \chi(n_{k+1}^2) \cdots \chi(n_{k+\ell}^2) \rangle_D \\ =& \begin{cases} \prod_{\substack{p \mid \cube, p \nmid N_E \\ p \equiv 1 \,(\mathrm{mod}\, 3)}} \left(1 + \frac{2}{p}\right)^{-1} & n_1 \cdots n_k n_{k+1}^2 \cdots n_{k+\ell}^2 = \cube, \\ o(1) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Assume that n is a cube and write $n = m^3$. Then $\chi(m^3) = 1$ for $(d, m) = 1$ and 0 if $(d, m) > 1$. We have

$$\sum_{\chi \in \mathcal{F}_{N_E}(D)} \chi(m^3) = \sum_{\chi \in \mathcal{F}_{m N_E}(D)} 1,$$

and the result follows by Corollary 3.2. Assume now that n is not a cube. For any $a, b \in \mathbb{Z}[\xi_3]$, we use the notation

$$\left(\frac{a}{b}\right)_3$$

to denote the cubic residue symbol defined over the Eisenstein ring $\mathbb{Z}[\xi_3]$. If $a, b \in \mathbb{Z}$, by cubic reciprocity (since integers are primary Eisenstein integers), we have that

$$\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3.$$

Using cubic reciprocity, we write the generating series for

$$\sum_{\chi \in \mathcal{F}_{N_E}(D)} \chi(n)$$

as

$$\prod_{\substack{p \equiv 1 \,(\mathrm{mod}\, 3) \\ p = \pi \overline{\pi}}} \left(1 + \frac{\left(\frac{\pi}{n}\right)_3}{p^s} + \frac{\overline{\left(\frac{\pi}{n}\right)_3}}{p^s}\right) = L(s, \chi_n) L(s, \overline{\chi_n}) F(s),$$

where $F(s)$ is analytic for $\mathrm{Re}(s) > 1/2 + \varepsilon$. We recall that

$$L(s, \chi_n) = \left(1 - \frac{\chi_n(3)}{3^s}\right) \prod_{p \equiv 1 \,(\mathrm{mod}\, 3)} \left(1 - \frac{\chi_n(\pi)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\chi_n}(\pi)}{p^s}\right)^{-1} \prod_{p \equiv 2 \,(\mathrm{mod}\, 3)} \left(1 - \frac{\chi_n(p)}{p^{2s}}\right)^{-1}$$

and is analytic for all $s \in \mathbb{C}$. This shows the result.

$\square$

Let

$$\delta_p = \begin{cases} \frac{2}{p+2} & p \equiv 1 \pmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing the product of the characters by its average, we replace $M(1/2; \alpha_1, \dots, \alpha_{k+\ell})$ by

$$\prod_{j=1}^{k} \mathcal{X}(1/2 + \alpha_j, E, d)^{-1/2} \prod_{j=1}^{\ell} \mathcal{X}(1/2 - \alpha_{j+k}, E, d)^{-1/2} R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}),$$

where

$$R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}) = \prod_p R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}; p)$$

where for $p \nmid N_E$,

$$\begin{aligned} &R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}; p) \\ &= 1 + (1 - \delta_p) \sum_{j=1}^{\infty} \sum_{e_1 + \cdots + e_k + 2e_{k+1} + \cdots + 2e_{k+\ell} = 3j} \prod_{h=1}^{k} \frac{a_{p^{e_h}}}{p^{e_h(1/2+\alpha_h)}} \prod_{h=k+1}^{k+\ell} \frac{a_{p^{e_h}}}{p^{e_h(1/2-\alpha_h)}} \end{aligned} \tag{25}$$

and for $p \mid N_E$,

$$R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}; p) = 1 + \sum_{j=1}^{\infty} \sum_{e_1 + \cdots + e_k + 2e_{k+1} + \cdots + 2e_{k+\ell} = 3j} \prod_{h=1}^{k} \frac{a_{p^{e_h}}}{p^{e_h(1/2+\alpha_h)}} \prod_{h=k+1}^{k+\ell} \frac{a_{p^{e_h}}}{p^{e_h(1/2-\alpha_h)}}. \tag{26}$$

In order to evaluate the above expression, we have to determine the poles when the shifts α_i tend to zero. Let

$$\begin{aligned} &G(\alpha_1, \dots, \alpha_{k+\ell}) := \\ &\prod_{j=1}^{k} X(1/2 + \alpha_j, E)^{-1/2} \prod_{j=1}^{\ell} X(1/2 - \alpha_{j+k}, E)^{-1/2} R_{E,k,\ell}(1/2, \alpha_1, \dots, \alpha_{k+\ell}), \end{aligned} \tag{27}$$

where we recall that $X(s, E)$ is defined by (19). Looking at the Euler factors (25) and (26), the poles come from the terms where $j = 1$, $e_h = 1$ for exactly one value of h, and $e_{k+m} = 1$ for exactly one value of m, and the rest of the e_i are zero, and this is for all the possible combinations $(h, k+m)$ with $1 \le h \le k$ and $1 \le m \le \ell$. Then, we rewrite $G(\alpha_1, \dots, \alpha_{k+\ell})$

as

$$G(\alpha_1,\dots,\alpha_{k+\ell}) = F(\alpha_1,\dots,\alpha_{k+\ell})\prod_{h=1}^{k}\prod_{m=1}^{\ell}\zeta(1+\alpha_h-\alpha_{k+m})$$
$$= \prod_{j=1}^{k} X(1/2+\alpha_j,E)^{-1/2}\prod_{j=1}^{\ell} X(1/2-\alpha_{j+k},E)^{-1/2} A_{E,k,\ell}(\alpha_1,\dots,\alpha_{k+\ell})$$
$$\times\prod_{h=1}^{k}\prod_{m=1}^{\ell}\zeta(1+\alpha_h-\alpha_{k+m}), \tag{28}$$

where

$$A_{E,k,\ell}(\alpha_1,\dots,\alpha_{k+\ell};p) = \prod_{h=1}^{k}\prod_{m=1}^{\ell}\left(1-\frac{1}{p^{1+\alpha_h-\alpha_{k+m}}}\right) R_{E,k,\ell}(1/2,\alpha_1,\dots,\alpha_{k+\ell};p). \tag{29}$$

Replacing in (24), we conjecture that

(30)

$$\left\langle Z(1/2,E,\chi)^k Z(1/2,E,\overline{\chi})^\ell\right\rangle_D^{(\alpha)} = \left\langle \sum_{\sigma\in\Xi_{k,\ell}} d^{\sum_{j=1}^{k}\alpha_{\sigma(j)}-\sum_{j=k+1}^{k+\ell}\alpha_{\sigma(j)}} G(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k+\ell)})\right\rangle_D .$$

Recall Lemma 2.5.3 from [CFK+05].

Lemma 4.3. *Suppose that $F_1(a_1,\dots,a_k,b_1,\dots,b_\ell)$ is symmetric in the variables a_j and in the variables b_j and is regular near $(0,\dots,0)$. Suppose that $f(s)=\frac{1}{s}+c+\cdots$ and let*

$$G_1(a_1,\dots,a_k,b_1,\dots,b_\ell) = F_1(a_1,\dots,b_\ell)\prod_{i=1}^{k}\prod_{j=1}^{\ell} f(a_i-b_j).$$

If for all i,j, we have $\alpha_i-\alpha_{k+j}$ is contained in the region of analyticity of $f(s)$ then Then

$$\sum_{\sigma\in\Xi_{k,\ell}} G_1(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)},\alpha_{\sigma(k+1)},\dots,\alpha_{\sigma(k+\ell)})$$
$$= \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!}\sum_{\tau\in\mathrm{Sym}(k+\ell)}\mathrm{Res}_{(z_1,\dots,z_{k+\ell})=(\alpha_{\tau(1)},\dots,\alpha_{\tau(k+\ell)})}\frac{G_1(z_1,\dots,z_{k+\ell})\Delta(z_1,\dots,z_{k+\ell})^2}{\prod_{h=1}^{k+\ell}\prod_{m=1}^{k+\ell}(z_h-\alpha_m)}$$
$$= \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!}\frac{1}{(2\pi i)^{k+\ell}}\oint\cdots\oint\frac{G_1(z_1,\dots,z_{k+\ell})\Delta(z_1,\dots,z_{k+\ell})^2}{\prod_{h=1}^{k+\ell}\prod_{m=1}^{k+\ell}(z_h-\alpha_m)}dz_1\cdots dz_{k+\ell},$$

where we are integrating over small circles around $\alpha_1,\dots,\alpha_{k+\ell}$.

Applying Lemma 4.3 in (30) for

$$G_1(\alpha_1,\dots,\alpha_{k+\ell}) = d^{\sum_{j=1}^{k}\alpha_j-\sum_{j=k+1}^{k+\ell}\alpha_j} G(\alpha_1,\dots,\alpha_{k+\ell})$$

we conjecture that

$$\left\langle Z(1/2,E,\chi)^k Z(1/2,E,\overline{\chi})^\ell\right\rangle_D^{(\alpha)} = \left\langle \Upsilon_{k,\ell}^{(\alpha)}(2\log d)\right\rangle_D ,$$

where

$$\Upsilon_{k,\ell}^{(\alpha)}(x) = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint \frac{G(z_1,\dots,z_{k+\ell})\Delta(z_1,\dots,z_{k+\ell})^2}{\prod_{h=1}^{k+\ell}\prod_{m=1}^{k+\ell}(z_h-\alpha_m)} e^{\frac{x}{2}\left(\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}\right)} dz_1 \dots dz_{k+\ell}.$$

This is in accordance to Theorem 2.3, where x is the conductor of $L(s,E,\chi)$, which is $\log(N_E d^2)$. Since N_E is constant, we use $x = N = 2\log d$.

Now we let $\alpha_i \to 0$ in order to obtain the conjecture stated at the beginning of the section, recalling that we can replace Z by L in light of (22).

5. The first two coefficients for k, ℓ positive integers

We show in this section that

$$\begin{aligned}\langle \Upsilon_{k,\ell}(2\log d)\rangle_D =& g_{k,\ell} A_{E,k,\ell}(0,\dots,0)\frac{2^{k\ell}}{(k\ell)!}\langle \log^{k\ell} d\rangle_D \\ &+ g_{k,\ell}\left(\gamma(k+\ell-2)+\log\left(\frac{N_E}{4\pi^2}\right)\right) A_{E,k,\ell}(0,\dots,0)\frac{2^{k\ell-1}}{(k\ell-1)!}\langle \log^{k\ell-1} d\rangle_D \\ &+ g_{k,\ell}\left(\frac{\partial A_{E,k,\ell}}{\partial z_k}(0,\dots,0) - \frac{\partial A_{E,k,\ell}}{\partial z_{k+\ell}}(0,\dots,0)\right)\frac{2^{k\ell-1}}{(k\ell-1)!}\langle \log^{k\ell-1} d\rangle_D \\ &+ O\left(\log^{k\ell-2} D\right). \end{aligned} \tag{31}$$

It is not hard to see that $\Upsilon_{k,\ell}(x)$ is a polynomial of degree (at most) $k\ell$, and we compute the coefficients of $x^{k\ell}$ and $x^{k\ell-1}$. To compute the coefficient of $x^{k\ell}$, we adapt the proof of [CFK+05] to the case $\ell \neq k$. In doing so, we consider a more general integral (Lemma 5.2) that allows us to go one step further and compute a general formula the coefficient of $x^{k\ell-1}$.

Conjecture 1.1 will then follow using the summation formulas of Section 3 in (31).

We now proceed to prove (31). After making the change of variables $z_i \to \frac{z_i}{x/2}$, and writing $\zeta(1+s) = 1/s + \gamma + \cdots$, we get

$$\begin{aligned}
\Upsilon_{k,\ell}(x) &= \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint \frac{F\left(\frac{z_1}{x/2}, \ldots, \frac{z_{k+\ell}}{x/2}\right) \prod_{h=1}^{k} \prod_{m=1}^{\ell} \zeta\left(1 + \frac{z_h}{x/2} - \frac{z_{k+m}}{x/2}\right)}{(z_1 \cdots z_{k+\ell})^{k+\ell}} \\
&\quad \times \Delta(z_1, \ldots, z_{k+\ell})^2 e^{\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}} dz_1 \ldots dz_{k+\ell} \\
&= \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint F\left(\frac{z_1}{x/2}, \ldots, \frac{z_{k+\ell}}{x/2}\right) \left(\frac{x}{2}\right)^{k\ell} \\
&\quad \times \left(1 + \frac{\gamma}{x/2}\left(\ell \sum_{h=1}^{k} z_h - k \sum_{m=1}^{\ell} z_{k+m}\right) + O(x^{-2})\right) \\
&\quad \times \frac{\Delta(z_1, \ldots, z_{k+\ell})^2}{\prod_{h=1}^{k} \prod_{m=1}^{\ell} (z_h - z_{k+m})(z_1 \cdots z_{k+\ell})^{k+\ell}} e^{\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}} dz_1 \ldots dz_{k+\ell} \\
&= \frac{(-1)^{(k+\ell)(k+\ell-1)/2} x^{k\ell}}{2^{k\ell} k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint F\left(\frac{z_1}{x/2}, \ldots, \frac{z_{k+\ell}}{x/2}\right) \\
&\quad \times \left(1 + \frac{2\gamma}{x}\left(\ell \sum_{h=1}^{k} z_h - k \sum_{m=1}^{\ell} z_{k+m}\right) + O(x^{-2})\right) \\
&\quad \times \frac{\Delta(z_1, \ldots, z_{k+\ell})\Delta(z_1, \ldots, z_k)\Delta(z_{k+1}, \ldots, z_{k+\ell})}{(z_1 \cdots z_{k+\ell})^{k+\ell}} e^{\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}} dz_1 \ldots dz_{k+\ell}.
\end{aligned}$$

This shows that $\Upsilon_{k,\ell}(x)$ is a polynomial of degree $k\ell$. The following lemma allows us how to extract the coefficients from the integrals.

Lemma 5.1. *Let $H(z_1, \ldots, z_h)$ be a function that is analytic around $(z_1, \ldots, z_h) = (0, \ldots, 0)$, and let $L_1(z_1, \ldots, z_h)$, $L_2(z_1, \ldots, z_h)$, and $P(z_1, \ldots, z_h)$ be polynomials.*

(a)

$$\begin{aligned}
&\lim_{x \to \infty} \frac{1}{(2\pi i)^h} \oint \cdots \oint H(z_1/x, \ldots, z_h/x) \left(1 + \frac{L_1(z_1, \ldots, z_h)}{x} + O(x^{-2})\right) \\
&\times \frac{P(z_1, \ldots, z_h)}{(z_1 \cdots z_h)^h} e^{L_2(z_1, \ldots, z_h)} dz_1 \ldots dz_h \\
=&\frac{H(0, \ldots, 0)}{(2\pi i)^h} \oint \cdots \oint \frac{P(z_1, \ldots, z_h)}{(z_1 \cdots z_h)^h} e^{L_2(z_1, \ldots, z_h)} dz_1 \ldots dz_h
\end{aligned}$$

(b)

$$\lim_{x\to\infty}\frac{x}{(2\pi i)^h}\oint\cdots\oint\left(H(z_1/x,\ldots,z_h/x)-H(0,\ldots,0)\right)\left(1+\frac{L_1(z_1,\ldots,z_h)}{x}+O(x^{-2})\right)$$
$$\times\frac{P(z_1,\ldots,z_h)}{(z_1\cdots z_h)^h}e^{L_2(z_1,\ldots,z_h)}dz_1\ldots dz_h$$
$$=\frac{H(0,\ldots,0)}{(2\pi i)^h}\oint\cdots\oint L_1(z_1,\ldots,z_h)\frac{P(z_1,\ldots,z_h)}{(z_1\cdots z_h)^h}e^{L_2(z_1,\ldots,z_h)}dz_1\ldots dz_h$$
$$+\sum_{j=1}^{h}\frac{\partial H}{\partial z_j}(0,\ldots,0)\frac{1}{(2\pi i)^h}\oint\cdots\oint\frac{z_jP(z_1,\ldots,z_h)}{(z_1\cdots z_h)^h}e^{L_2(z_1,\ldots,z_h)}dz_1\ldots dz_h.$$

Proof. (a) The term involving $L_1(z_1,\ldots,z_h)$ goes to zero when x goes to ∞. The same is true for $O(x^{-2})$. We note that the term $H(z_1/x,\ldots,z_h/x)$ will tend to $H(0,\ldots,0)$ in the limit and can be extracted as a constant.

(b) The term involving $L_1(z_1,\ldots,z_h)$ contributes because of the cancelation of x and $\frac{1}{x}$. Then there is another possible contribution coming from first derivatives of $H(z_1/x,\ldots,z_h/x)$. More precisely, we have

$$\lim_{x\to\infty}\frac{x}{(2\pi i)^h}\oint\cdots\oint\left(H(z_1/x,\ldots,z_h/x)-H(0,\ldots,0)\right)\frac{P(z_1,\ldots,z_h)}{(z_1\cdots z_h)^h}e^{L_2(z_1,\ldots,z_h)}dz_1\ldots dz_h$$
$$=\frac{1}{(2\pi i)^h}\oint\cdots\oint\lim_{x\to\infty}\frac{H(z_1/x,\ldots,z_h/x)-H(0,\ldots,0)}{1/x}\frac{P(z_1,\ldots,z_h)}{(z_1\cdots z_h)^h}e^{L_2(z_1,\ldots,z_h)}dz_1\ldots dz_h.$$

By considering the Taylor expansion we have

$$\frac{H(z_1/x,\ldots,z_h/x)-H(0,\ldots,0)}{1/x}=x\sum_{j=1}^{h}\frac{z_j}{x}\frac{\partial H}{\partial z_j}(0,\ldots,0).$$

Since the terms $\frac{\partial H}{\partial z_j}(0,\ldots,0)$ are constant, they can be extracted from the integral. □

We now proceed to the computation, following the technique from (2.7.11) in [CFK+05]. By Lemma 5.1 (a), the main coefficient of the polynomial $\Upsilon_{k,\ell}(x)$ is given by $g^0_{k,\ell}F(0,\ldots,0)$, where $g^0_{k,\ell}$ is a constant arising from the integral. Expanding the Vandermonde determinant (12), we proceed as follows.

$$g^0_{k,\ell}:=\lim_{x\to\infty}\frac{\Upsilon_{k,\ell}(x)}{F(0,\ldots,0)x^{k\ell}}=\frac{(-1)^{(k+\ell)(k+\ell-1)/2+k\ell}}{2^{k\ell}k!\ell!(2\pi i)^{k+\ell}}\oint\cdots\oint e^{\sum_{h=1}^{k}z_h-\sum_{m=1}^{\ell}z_{k+m}}$$
$$\times\left(\sum_\sigma\operatorname{sgn}(\sigma)z_1^{\sigma(0)}\cdots z_k^{\sigma(k-1)}z_{k+1}^{\sigma(k)}\cdots z_{k+\ell}^{\sigma(k+\ell-1)}\right)\left(\sum_\tau\operatorname{sgn}(\tau)z_1^{\tau(0)}\cdots z_k^{\tau(k-1)}\right)$$
$$\times\left(\sum_\rho\operatorname{sgn}(\rho)z_{k+1}^{\rho(0)}\cdots z_{k+\ell}^{\rho(\ell-1)}\right)(z_1\cdots z_{k+\ell})^{-(k+\ell)}dz_1\cdots dz_{k+\ell}.$$

Here σ, τ, and ρ are permutations of $\{0,\ldots,k+\ell-1\}$, $\{0,\ldots,k-1\}$ and $\{0,\ldots,\ell-1\}$ respectively. Since the integrand is symmetric with respect to the $z_1,\ldots,z_k$ and with respect to the $z_{k+1},\ldots,z_{k+\ell}$, we can permute the variables $z_1,\ldots,z_k$ so that z_j appears with exponent $j-1$ in the sum over τ. This redefines the permutations inside the sum over σ and changes the sign by canceling the $\mathrm{sgn}(\tau)$. We can do the same over ρ. We are left with $k!\ell!$ copies of the sum over the permutations σ, and then

$$
\begin{aligned}
g^0_{k,\ell} = & \frac{(-1)^{(k+\ell)(k+\ell-1)/2+k\ell}}{2^{k\ell}(2\pi i)^{k+\ell}} \oint \cdots \oint e^{\sum_{h=1}^{k} z_h - \sum_{m=1}^{\ell} z_{k+m}} \\
& \times \Bigg(\sum_{\sigma} \mathrm{sgn}(\sigma) z_1^{-(k+\ell-\sigma(0))} z_2^{-(k+\ell-\sigma(1)-1)} \cdots z_k^{-(k+\ell-\sigma(k-1)-(k-1))} \\
& z_{k+1}^{-(k+\ell-\sigma(k))} z_{k+2}^{-(k+\ell-\sigma(k+1)-1)} \cdots z_{k+\ell}^{-(k+\ell-\sigma(k+\ell-1)-(\ell-1))} \Bigg) dz_1 \cdots dz_{k+\ell}.
\end{aligned}
$$

Instead of computing $g^0_{k,\ell}$, we will consider a more general integral. This will help us compute the coefficient of $x^{k\ell-1}$ later.

Lemma 5.2. *Let*

$$
\begin{aligned}
\mathfrak{g}_{k,\ell}(u,v) := & \frac{(-1)^{(k+\ell)(k+\ell-1)/2+k\ell}}{2^{k\ell}(2\pi i)^{k+\ell}} \oint \cdots \oint e^{u\sum_{h=1}^{k} z_h - v\sum_{m=1}^{\ell} z_{k+m}} \\
& \times \Bigg(\sum_{\sigma} \mathrm{sgn}(\sigma) z_1^{-(k+\ell-\sigma(0))} z_2^{-(k+\ell-\sigma(1)-1)} \cdots z_k^{-(k+\ell-\sigma(k-1)-(k-1))} \\
& z_{k+1}^{-(k+\ell-\sigma(k))} z_{k+2}^{-(k+\ell-\sigma(k+1)-1)} \cdots z_{k+\ell}^{-(k+\ell-\sigma(k+\ell-1)-(\ell-1))} \Bigg) dz_1 \cdots dz_{k+\ell}.
\end{aligned}
$$

Then we have

$$
\mathfrak{g}_{k,\ell}(u,v) = \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \left(\frac{u+v}{2} \right)^{k\ell}.
$$

We remark that $g^0_{k,\ell} = \mathfrak{g}_{k,\ell}(1,1)$.

Proof. Since

$$
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C (-t)^{-z} e^{-t} (-dt),
$$

where the path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction, and returns to the starting point, the integral defining $\mathfrak{g}_{k,\ell}(u,v)$

can be rewritten as

$$\begin{aligned}\mathfrak{g}_{k,\ell}(u,v) =& \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{2^{k\ell}} \sum_\sigma \operatorname{sign}(\sigma) u^{k+\ell-\sigma(0)-1} u^{k+\ell-\sigma(1)-2} \cdots u^{k+\ell-\sigma(k-1)-k} \\ & \times v^{k+\ell-\sigma(k)-1} v^{k+\ell-\sigma(k+1)-2} \cdots v^{k+\ell-\sigma(k+\ell-1)-\ell} \\ & \times \Big(\Gamma(k+\ell-\sigma(0))\, \Gamma(k+\ell-\sigma(1)-1) \cdots \Gamma(k+\ell-\sigma(k-1)-(k-1)) \\ & \times (-1)^{\sigma(k)} \Gamma(k+\ell-\sigma(k))\, (-1)^{\sigma(k+1)+1} \Gamma(k+\ell-\sigma(k+1)-1) \cdots \\ & \times (-1)^{\sigma(k+\ell-1)+\ell-1} \Gamma(k+\ell-\sigma(k+\ell-1)-(\ell-1)) \Big)^{-1} \\ =& \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{2^{k\ell}} \\ & \times \begin{vmatrix} \frac{u^{\ell+k-1}}{\Gamma(\ell+k)} & \frac{u^{\ell+k-2}}{\Gamma(\ell+k-1)} & \cdots & \frac{u^{\ell}}{\Gamma(\ell+1)} & \frac{v^{k+\ell-1}}{\Gamma(k+\ell)} & \frac{-v^{k+\ell-2}}{\Gamma(k+\ell-1)} & \cdots & \frac{(-1)^{\ell-1}v^{k}}{\Gamma(k+1)} \\ \frac{u^{\ell+k-2}}{\Gamma(\ell+k-1)} & \frac{u^{\ell+k-3}}{\Gamma(\ell+k-2)} & \cdots & \frac{u^{\ell-1}}{\Gamma(\ell)} & \frac{-v^{k+\ell-2}}{\Gamma(k+\ell-1)} & \frac{v^{k+\ell-3}}{\Gamma(k+\ell-2)} & \cdots & \frac{(-1)^{\ell}v^{k-1}}{\Gamma(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{u^{0}}{\Gamma(1)} & \frac{u^{-1}}{\Gamma(0)} & \cdots & \frac{u^{1-k}}{\Gamma(2-k)} & \frac{(-1)^{k+\ell-1}v^{0}}{\Gamma(1)} & \frac{(-1)^{k+\ell}v^{-1}}{\Gamma(0)} & \cdots & \frac{(-1)^{k+2\ell-2}v^{1-\ell}}{\Gamma(2-\ell)} \end{vmatrix}.\end{aligned}$$

Notice that $\frac{1}{\Gamma(-k)} = 0$ for $k = 0, 1, \ldots$. We proceed by multiplying the first row by $(\ell+k-1)!$, the second row by $(\ell+k-2)!$ and so on. We divide the first column by $0!$, the second column by $1!$, up to the k column by $(k-1)!$, then the $k+1$ column by $0!$, the $k+2$ column by $1!$ until the $k+\ell$ column by $(\ell-1)!$. We get

$$\mathfrak{g}_{k,\ell}(u,v) = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \times$$

$$\times \begin{vmatrix} \binom{k+\ell-1}{0} u^{k+\ell-1} & \cdots & \binom{k+\ell-1}{k-1} u^{\ell} & \binom{k+\ell-1}{0} v^{k+\ell-1} & -\binom{k+\ell-1}{1} v^{k+\ell-2} & \cdots & (-1)^{\ell-1}\binom{k+\ell-1}{\ell-1} v^{k} \\ \binom{k+\ell-2}{0} u^{k+\ell-2} & \cdots & \binom{k+\ell-2}{k-1} u^{\ell-1} & -\binom{k+\ell-2}{0} v^{k+\ell-2} & \binom{k+\ell-2}{1} v^{k+\ell-3} & \cdots & (-1)^{\ell}\binom{k+\ell-2}{\ell-1} v^{k-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{0}{0} u^{0} & \cdots & \binom{0}{k-1} u^{1-k} & (-1)^{k+\ell-1}\binom{0}{0} v^{0} & (-1)^{k+\ell}\binom{0}{1} v^{-1} & \cdots & (-1)^{k+2\ell-2}\binom{0}{\ell-1} v^{1-\ell} \end{vmatrix}.$$

Notice that the terms of the form $\binom{a}{b}$ with $a < b$ are equal to zero.

We reverse the order of the rows. We also multiply the columns $k+1, k+2, \ldots, k+\ell$ by $(-1)^{k+\ell-1}$. We get

$$\mathfrak{g}_{k,\ell}(u,v) = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} (-1)^{(k+\ell)(k+\ell-1)/2} (-1)^{(k+\ell-1)\ell} \times \tag{32}$$

$$\begin{vmatrix} \binom{0}{0} u^{0} & \cdots & \binom{0}{k-1} u^{1-k} & \binom{0}{0} v^{0} & -\binom{0}{1} v^{-1} & \cdots & (-1)^{\ell-1}\binom{0}{\ell-1} v^{1-\ell} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{k+\ell-2}{0} u^{k+\ell-2} & \cdots & \binom{k+\ell-2}{k-1} u^{\ell-1} & (-1)^{k+\ell-2}\binom{k+\ell-2}{0} v^{k+\ell-2} & (-1)^{k+\ell-1}\binom{k+\ell-2}{1} v^{k+\ell-3} & \cdots & (-1)^{k+2\ell-3}\binom{k+\ell-2}{\ell-1} v^{k-1} \\ \binom{k+\ell-1}{0} u^{k+\ell-1} & \cdots & \binom{k+\ell-1}{k-1} u^{\ell} & (-1)^{k+\ell-1}\binom{k+\ell-1}{0} v^{k+\ell-1} & (-1)^{k+\ell}\binom{k+\ell-1}{1} v^{k+\ell-2} & \cdots & (-1)^{k+2\ell-2}\binom{k+\ell-1}{\ell-1} v^{k} \end{vmatrix}.$$

Now, the first k columns of the above matrix are the same as the first k columns of

$$M(u) = \begin{pmatrix} \binom{0}{0}u^0 & \binom{0}{1}u^{-1} & \cdots & \binom{0}{k+\ell-1}u^{1-k-\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+\ell-1}{0}u^{k+\ell-1} & \binom{k+\ell-1}{1}u^{k+\ell-2} & \cdots & \binom{k+\ell-1}{k+\ell-1}u^0 \end{pmatrix},$$

which is a lower triangular matrix of determinant 1. Its inverse is $N(u)$, where

$$N(v) = \begin{pmatrix} \binom{0}{0}v^0 & -\binom{0}{1}v^{-1} & \cdots & (-1)^{k+\ell-1}\binom{0}{k+\ell-1}v^{1-k-\ell} \\ -\binom{1}{0}v^1 & \binom{1}{1}v^0 & \cdots & (-1)^{k+\ell}\binom{1}{k+\ell-1}v^{1-k\ell} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{k+\ell-1}\binom{k+\ell-1}{0}v^{k+\ell-1} & (-1)^{k+\ell}\binom{k+\ell-1}{1}v^{k+\ell-2} & \cdots & \binom{k+\ell-1}{k+\ell-1}v^0 \end{pmatrix}.$$

The columns $k+1,\dots,k+\ell$ in (32) are the same as the first ℓ columns of the above matrix.

We multiply (32) by $\det M(v)$ and this does not change its value. Thus,

$$\begin{aligned}
\mathfrak{g}_{k,\ell}(u,v) =& \frac{(-1)^{k\ell}}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \\
&\times \begin{vmatrix} \binom{0}{0}v^0 & \binom{0}{1}v^{-1} & \cdots & \binom{0}{k+\ell-1}v^{1-k-\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+\ell-1}{0}v^{k+\ell-1} & \binom{k+\ell-1}{1}v^{k+\ell-2} & \cdots & \binom{k+\ell-1}{k+\ell-1}v^0 \end{vmatrix} \\
&\times \begin{vmatrix} \binom{0}{0}u^0 & \cdots & \binom{0}{k-1}u^{1-k} & \binom{0}{0}v^0 & \cdots & (-1)^{\ell-1}\binom{0}{\ell-1}v^{1-\ell} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \binom{k+\ell-2}{0}u^{k+\ell-2} & \cdots & \binom{k+\ell-2}{k-1}u^{\ell-1} & (-1)^{k+\ell-2}\binom{k+\ell-2}{0}v^{k+\ell-2} & \cdots & (-1)^{k+2\ell-3}\binom{k+\ell-2}{\ell-1}v^{k-1} \\ \binom{k+\ell-1}{0}u^{k+\ell-1} & \cdots & \binom{k+\ell-1}{k-1}u^{\ell} & (-1)^{k+\ell-1}\binom{k+\ell-1}{0}v^{k+\ell-1} & \cdots & (-1)^{k+2\ell-2}\binom{k+\ell-1}{\ell-1}v^{k} \end{vmatrix} \\
=& \frac{(-1)^{k\ell}}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \\
&\times \begin{vmatrix} (u+v)^0\binom{0}{0} & (u+v)^{-1}\binom{0}{1} & \cdots & (u+v)^{1-k}\binom{0}{k-1} & 1 & 0 & \cdots & 0 \\ (u+v)^1\binom{1}{0} & (u+v)^0\binom{1}{1} & \cdots & (u+v)^{2-k}\binom{1}{k-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (u+v)^{\ell-1}\binom{\ell-1}{0} & (u+v)^{\ell-2}\binom{\ell-1}{1} & \cdots & (u+v)^{\ell-k}\binom{\ell-1}{k-1} & 0 & 0 & \cdots & 1 \\ (u+v)^{\ell}\binom{\ell}{0} & (u+v)^{\ell-1}\binom{\ell}{1} & \cdots & (u+v)^{\ell+1-k}\binom{\ell}{k-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (u+v)^{k+\ell-1}\binom{k+\ell-1}{0} & (u+v)^{k+\ell-2}\binom{k+\ell-1}{1} & \cdots & (u+v)^{\ell}\binom{k+\ell-1}{k-1} & 0 & 0 & \cdots & 0 \end{vmatrix}.
\end{aligned}$$

Computing the determinant by blocks,

$$\begin{aligned}\mathfrak{g}_{k,\ell}(u,v) =& \frac{(-1)^{k\ell}}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \\ & \times (-1)^{k\ell} \begin{vmatrix} (u+v)^{\ell}\binom{\ell}{0} & (u+v)^{\ell-1}\binom{\ell}{1} & \cdots & (u+v)^{\ell+1-k}\binom{\ell}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ (u+v)^{k+\ell-1}\binom{k+\ell-1}{0} & (u+v)^{k+\ell-2}\binom{k+\ell-1}{1} & \cdots & (u+v)^{\ell}\binom{k+\ell-1}{k-1} \end{vmatrix}.\end{aligned}$$

Reversing the order of the rows,

$$\begin{aligned}\mathfrak{g}_{k,\ell}(u,v) =& \frac{1}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \\ & \times (-1)^{k(k-1)/2} \begin{vmatrix} (u+v)^{k+\ell-1}\binom{k+\ell-1}{0} & (u+v)^{k+\ell-2}\binom{k+\ell-1}{1} & \cdots & (u+v)^{\ell}\binom{k+\ell-1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ (u+v)^{\ell}\binom{\ell}{0} & (u+v)^{\ell-1}\binom{\ell}{1} & \cdots & (u+v)^{\ell+1-k}\binom{\ell}{k-1} \end{vmatrix}.\end{aligned}$$

Extracting the powers of $(u+v)$,

$$\begin{aligned}\mathfrak{g}_{k,\ell}(u,v) =& \frac{1}{2^{k\ell}} \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \\ & \times (-1)^{k(k-1)/2}(u+v)^{k\ell} \begin{vmatrix} \binom{k+\ell-1}{0} & \binom{k+\ell-1}{1} & \cdots & \binom{k+\ell-1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\ell}{0} & \binom{\ell}{1} & \cdots & \binom{\ell}{k-1} \end{vmatrix}.\end{aligned}$$

Now the matrix can be decomposed as

$$\begin{aligned}&\begin{pmatrix} \binom{k+\ell-1}{0} & \binom{k+\ell-1}{1} & \cdots & \binom{k+\ell-1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\ell}{0} & \binom{\ell}{1} & \cdots & \binom{\ell}{k-1} \end{pmatrix} \\ =& \begin{pmatrix} \binom{k-1}{0} & \binom{k-1}{1} & \cdots & \binom{k-1}{k-1} \\ \binom{k-2}{0} & \binom{k-2}{1} & \cdots & \binom{k-2}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} \end{pmatrix} \times \begin{pmatrix} \binom{\ell}{0} & \binom{\ell}{1} & \cdots & \binom{\ell}{k-1} \\ \binom{\ell}{-1} & \binom{\ell}{0} & \cdots & \binom{\ell}{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\ell}{1-k} & \binom{\ell}{2-k} & \cdots & \binom{\ell}{0} \end{pmatrix}.\end{aligned}$$

The first matrix on the right-hand side is left upper triangular, with determinant $(-1)^{k(k-1)/2}$. The second matrix on the right-hand side is right upper triangular with determinant 1. Thus we have

$$\mathfrak{g}_{k,\ell}(u,v) = \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} \left(\frac{u+v}{2}\right)^{k\ell}.$$

□

For the main coefficient, applying Lemma 5.1 (a), we set $u = v = 1$ and therefore

$$g_{k,\ell}^{0} = \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!}.$$

If we normalize this coefficient by multiplying by $(k\ell)!$ so that it becomes an integer, we recover $g_{k,\ell}$ from (1)

$$g_{k,\ell} = (k\ell)! g^0_{k,\ell}.$$

For the coefficient of $x^{k\ell-1}$, Lemma 5.1 (b) implies that this coefficient equals

$$g^1_{k,\ell} F(0,\ldots,0) + \sum_{j=1}^{k} g^{1,1}_{k,\ell,j} \frac{\partial F}{\partial z_j}(0,\ldots,0) - \sum_{m=1}^{\ell} g^{1,2}_{k,\ell,k+m} \frac{\partial F}{\partial z_m}(0,\ldots,0).$$

By Lemma 5.1 (b), we have

$$\begin{aligned} g^1_{k,\ell} :=& \frac{2\gamma(-1)^{(k+\ell)(k+\ell-1)/2+k\ell}}{2^{k\ell}k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint e^{\sum_{h=1}^k z_h - \sum_{m=1}^{\ell} z_{k+m}} \left(\ell \sum_{h=1}^{k} z_h - k \sum_{m=1}^{\ell} z_{k+m}\right) \\ &\times \frac{\Delta(z_1,\ldots,z_{k+\ell})\Delta(z_1,\ldots,z_k)\Delta(z_{k+1},\ldots,z_{k+\ell})}{(z_1\cdots z_{k+\ell})^{k+\ell}} dz_1 \ldots dz_{k+\ell} \\ =& 2\gamma\ell \frac{\partial \mathfrak{g}_{k,\ell}}{\partial u}(1,1) + 2\gamma k \frac{\partial \mathfrak{g}_{k,\ell}}{\partial v}(1,1) \\ =& \gamma \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} k\ell(k+\ell). \end{aligned}$$

For the $g^{1,1}_{k,\ell,j}, g^{1,2}_{k,\ell,k+m}$, since $\frac{\partial F}{\partial z_1}(0,\ldots,0) = \cdots = \frac{\partial F}{\partial z_k}(0,\ldots,0)$ and $\frac{\partial F}{\partial z_{k+1}}(0,\ldots,0) = \cdots = \frac{\partial F}{\partial z_{k+\ell}}(0,\ldots,0)$, it suffices to compute

$$\begin{aligned} \sum_{j=1}^{k} g^{1,1}_{k,\ell,j} =& \frac{2(-1)^{(k+\ell)(k+\ell-1)/2}}{2^{k\ell}k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint e^{\sum_{j=1}^k z_j - \sum_{m=1}^{\ell} z_{k+m}} \left(\sum_{j=1}^{k} z_j\right) \\ &\times \frac{\Delta(z_1,\ldots,z_{k+\ell})\Delta(z_1,\ldots,z_k)\Delta(z_{k+1},\ldots,z_{k+\ell})}{(z_1\cdots z_{k+\ell})^{k+\ell}} dz_1 \ldots dz_{k+\ell} \\ =& 2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial u}(1,1) \\ =& \frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} k\ell. \end{aligned}$$

We remark that the factor of 2 in front of the integral comes from the fact that we have $F\left(\frac{z_1}{x/2},\ldots,\frac{z_{k+\ell}}{x/2}\right)$ instead of $F\left(\frac{z_1}{x},\ldots,\frac{z_{k+\ell}}{x}\right)$ inside the integral and that yields a factor of 2 upon differentiation and application of Lemma 5.1 (b).

Analogously,

$$\sum_{m=1}^{\ell} g^{1,2}_{k,\ell,k+m} = \frac{(-1)^{(k+\ell)(k+\ell-1)/2}}{k!\ell!(2\pi i)^{k+\ell}} \oint \cdots \oint e^{\sum_{j=1}^{k} z_j - \sum_{m=1}^{\ell} z_{k+m}} \left(-\sum_{m=1}^{\ell} z_{k+m}\right)$$
$$\times \frac{\Delta(z_1,\dots,z_{k+\ell})\Delta(z_1,\dots,z_k)\Delta(z_{k+1},\dots,z_{k+\ell})}{(z_1\cdots z_{k+\ell})^{k+\ell}} dz_1 \dots dz_{k+\ell}$$
$$=2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial v}(1,1)$$
$$=\frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} k\ell.$$

Putting everything together, and recalling from (28) that

$$F(\alpha_1,\dots,\alpha_{k+\ell}) = \prod_{j=1}^{k} X(1/2+\alpha_j,E)^{-1/2} \prod_{j=1}^{\ell} X(1/2-\alpha_{j+k},E)^{-1/2} A_{E,k,\ell}(\alpha_1,\dots,\alpha_{k+\ell}),$$

the contribution from the derivatives equals

$$2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial u}(1,1)\frac{\partial F}{\partial z_k}(0,\dots,0) - 2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial v}(1,1)\frac{\partial F}{\partial z_{k+\ell}}(0,\dots,0)$$
$$=2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial u}(1,1)\frac{\partial A_{E,k,\ell}}{\partial z_k}(0,\dots,0) - 2\frac{\partial \mathfrak{g}_{k,\ell}}{\partial v}(1,1)\frac{\partial A_{E,k,\ell}}{\partial z_{k+\ell}}(0,\dots,0)$$
$$- A_{E,k,\ell}(0,\dots,0)\left(\frac{\partial \mathfrak{g}_{k,\ell}}{\partial u}(1,1)X'(1/2,E) - \frac{\partial \mathfrak{g}_{k,\ell}}{\partial v}(1,1)X'(1/2,E)\right)$$
$$=\frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} k\ell \left(\frac{\partial A_{E,k,\ell}}{\partial z_k}(0,\dots,0) - \frac{\partial A_{E,k,\ell}}{\partial z_{k+\ell}}(0,\dots,0)\right)$$
$$+ A_{E,k,\ell}(0,\dots,0)\frac{\prod_{h=0}^{k-1} h! \prod_{h=0}^{\ell-1} h!}{\prod_{h=0}^{k+\ell-1} h!} k\ell \left(-\gamma + \frac{1}{2}\log\left(\frac{N_E}{4\pi^2}\right)\right),$$

where we have used Lemma 4.1 in order to evaluate $X'(1/2,E)$.

We have then proven (31).

We remark that in the symplectic and orthogonal cases it suffices to differentiate

$$\mathfrak{g}_k(u) = \left(\frac{k(k+1)}{2}\right)! \prod_{h=1}^{k} \frac{h!}{(2h)!} u^{\frac{k(k+1)}{2}} \text{ and } \mathfrak{g}_k(u) = 2^{k-1}\left(\frac{k(k-1)}{2}\right)! \prod_{h=1}^{k-1} \frac{h!}{(2h)!} u^{\frac{k(k-1)}{2}}.$$

respect to u and do an analogous treatment to the previous page in order to get the second (and higher) geometric coefficients.

6. The case $\ell = 0$

In the previous session, we assume that both $k,\ell > 0$. Now we consider the case when $\ell = 0$. This case is different because $R_{E,k,0}(1/2,\alpha_1,\dots,\alpha_k)$ does not have poles when the shifts are zero.

Indeed, we have for $p \nmid N_E$,

$$R_{E,k,0}(1/2, \alpha_1, \dots, \alpha_k; p) = 1 + (1 - \delta_p) \sum_{j=1}^{\infty} \sum_{e_1 + \dots + e_k = 3j} \prod_{h=1}^{k} \frac{a_{p^{e_h}}}{p^{e_h(1/2+\alpha_h)}}$$

and for $p \mid N_E$,

$$R_{E,k,0}(1/2, \alpha_1, \dots, \alpha_k; p) = 1 + \sum_{j=1}^{\infty} \sum_{e_1 + \dots + e_k = 3j} \prod_{h=1}^{k} \frac{a_{p^{e_h}}}{p^{e_h(1/2+\alpha_h)}}.$$

We then work with the expression

$$G(\alpha_1, \dots, \alpha_k) := \prod_{j=1}^{k} X(1/2 + \alpha_j, E, d)^{-1/2} R_{E,k,0}(1/2, \alpha_1, \dots, \alpha_k),$$

and the conjecture from (30) is that

$$\left\langle Z(1/2, E, \chi)^k \right\rangle_D^{(\alpha)} = \left\langle d^{\sum_{j=1}^k \alpha_j} G(\alpha_1, \dots, \alpha_k) \right\rangle_D. \tag{33}$$

Now, we can take directly $\alpha_i \to 0$ as $G(0, \dots, 0) = A_{E,k,0}(0, \dots, 0)$ is defined, and therefore, we conjecture that as $D \to \infty$

$$\left\langle Z(1/2, E, \chi)^k \right\rangle_D \sim A_{E,k,0}(0, \dots, 0). \tag{34}$$

7. The sum of the first two terms

We prove in this section the formulas for the coefficients $c_{k\ell}$ and $c_{k\ell-1}$ in Conjecture 1.1. In doing our computations, we write the formulas in such a way to allow the generalization to $k, \ell \notin \mathbb{Z}$.

Combining equations (31) and (34) with Corollary 3.2, we have

$$\begin{aligned}
\langle \Upsilon_{k,\ell}(2 \log d) \rangle_D =& g_{k,\ell} A_{E,k,\ell}(0, \dots, 0) 2^{k\ell} \sum_{j=0}^{k\ell} \frac{(-1)^j \log^{k\ell-j} D}{(k\ell - j)!} \\
&+ g_{k,\ell} \left(\gamma(k + \ell - 2) + \log\left(\frac{N_E}{4\pi^2} \right) \right) A_{E,k,\ell}(0, \dots, 0) 2^{k\ell-1} \sum_{j=0}^{k\ell-1} \frac{(-1)^j \log^{k\ell-1-j} D}{(k\ell - 1 - j)!} \\
&+ g_{k,\ell} \left(\frac{\partial A_{E,k,\ell}}{\partial z_k}(0, \dots, 0) - \frac{\partial A_{E,k,\ell}}{\partial z_{k+\ell}}(0, \dots, 0) \right) 2^{k\ell-1} \sum_{j=0}^{k\ell-1} \frac{(-1)^j \log^{k\ell-1-j} D}{(k\ell - 1 - j)!} \\
&+ O\left(\log^{k\ell-2} D \right),
\end{aligned}$$

where it is understood that the second term is not present if $\ell = 0$.

By formula (29), the Euler factors of $A_{E,k,\ell}(z_1,\ldots,z_{k+\ell})$ can be rewritten using (25) and (26) as

$$A_{E,k,\ell}(z_1,\ldots,z_{k+\ell};p)=\prod_{h=1}^{k}\prod_{m=1}^{\ell}\left(1-\frac{1}{p^{1+z_h-z_{k+m}}}\right)(1-\gamma_p\delta_p)$$
$$\times\left(\frac{\gamma_p\delta_p}{1-\delta_p}+\frac{1}{3}\left(\prod_{h=1}^{k}\mathcal{L}_p\left(\frac{1}{p^{1/2+z_h}}\right)\prod_{m=1}^{\ell}\mathcal{L}_p\left(\frac{1}{p^{1/2-z_{k+m}}}\right)\right.\right.$$
$$\left.\left.+\prod_{h=1}^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2+z_h}}\right)\prod_{m=1}^{\ell}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2-z_{k+m}}}\right)+\prod_{h=1}^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2+z_h}}\right)\prod_{m=1}^{\ell}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2-z_{k+m}}}\right)\right)\right),$$

where ξ_3 is a primitive third root of unity,

$$\delta_p=\begin{cases}\frac{2}{p+2} & p\equiv 1\,(\mathrm{mod}\,3),\\ 0 & \text{otherwise},\end{cases}\qquad \gamma_p=\begin{cases}1 & p\nmid N_E,\\ 0 & p\mid N_E,\end{cases}$$

and the $\mathcal{L}_p(u)$ are the Euler factors of the L-function, namely

$$\mathcal{L}_p(u)=\begin{cases}(1-a_pu+u^2)^{-1} & p\nmid N_E,\\ (1-a_pu)^{-1} & p\mid N_E,\end{cases}$$

and a_p is normalized as in (14).

In order to simplify the notation we collapse the first k variables and the second ℓ variables into just two variables. This change of notation is crucial to extend the formulas for k,ℓ non-integral. Thus, we consider

(35)
$$A_{E,k,\ell}(z_1,z_2;p)=\left(1-\frac{1}{p^{1+z_1-z_2}}\right)^{k\ell}(1-\gamma_p\delta_p)\left(\frac{\gamma_p\delta_p}{1-\delta_p}+\frac{1}{3}\left(\mathcal{L}_p\left(\frac{1}{p^{1/2+z_1}}\right)^{k}\mathcal{L}_p\left(\frac{1}{p^{1/2-z_2}}\right)^{\ell}\right.\right.$$
$$\left.\left.+\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2+z_1}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2-z_2}}\right)^{\ell}+\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2+z_1}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2-z_2}}\right)^{\ell}\right)\right).$$

With this new convention, we obtain

$$\begin{aligned}\langle\Upsilon_{k,\ell}(2\log d)\rangle_D =&g_{k,\ell}A_{E,k,\ell}(0,0)2^{k\ell}\sum_{j=0}^{k\ell}\frac{(-1)^j\log^{k\ell-j}D}{(k\ell-j)!}\\ &+g_{k,\ell}\left(\gamma(k+\ell-2)+\log\left(\frac{N_E}{4\pi^2}\right)\right)A_{E,k,\ell}(0,0)2^{k\ell-1}\sum_{j=0}^{k\ell-1}\frac{(-1)^j\log^{k\ell-1-j}D}{(k\ell-1-j)!}\\ &+g_{k,\ell}\left(\frac{1}{k}\frac{\partial A_{E,k,\ell}}{\partial z_1}(0,0)-\frac{1}{\ell}\frac{\partial A_{E,k,\ell}}{\partial z_2}(0,0)\right)2^{k\ell-1}\sum_{j=0}^{k\ell-1}\frac{(-1)^j\log^{k\ell-1-j}D}{(k\ell-1-j)!}\\ &+O\left(\log^{k\ell-2}D\right).\end{aligned}\tag{36}$$

This gives Conjecture 1.1.

Now we focus on extending this construction for k and ℓ real. Recall from equation (4) in the introduction that Keating and Snaith [KS00] considered

$$\frac{G(k+1)G(\ell+1)}{G(k+\ell+1)}, \text{ for } \operatorname{Re}(k), \operatorname{Re}(\ell), \operatorname{Re}(k+\ell) > -1. \tag{37}$$

In our experiments with $k, \ell \in \mathbb{C}$ and satisfying the conditions of [KS00], we have encountered an additional difficulty in the cases where the term

$$L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell \tag{38}$$

is not defined when $L(1/2, E, \chi) = 0$. For example, this issue arises when $k + \ell$ is a negative real number or when either k or ℓ is purely imaginary. We have conducted numerical evaluations excluding these terms (so that we divide by the number of χ such that $L(1/2, E, \chi) \neq 0$), and the model still matches the numerical results.

The above discussion, together with Remark 8.1, allow us to extend formula (36) in the following way for $k, \ell \in \mathbb{C}$, $\operatorname{Re}(k), \operatorname{Re}(\ell)$, and $\operatorname{Re}(k+\ell) > -1$:

$$\begin{aligned} \langle \Upsilon_{k,\ell}(2\log d)\rangle_D \sim & \frac{G(k+1)G(\ell+1)}{G(k+\ell+1)} A_{E,k,\ell}(0,0) \frac{2^{k\ell}}{D} |\Gamma(k\ell+1, -\log D)| \\ & + \frac{G(k+1)G(\ell+1)}{G(k+\ell+1)} k\ell \left(\gamma(k+\ell-2) + \log\left(\frac{N_E}{4\pi^2}\right)\right) A_{E,k,\ell}(0,0) \frac{2^{k\ell-1}}{D} |\Gamma(k\ell, -\log D)| \\ & + \frac{G(k+1)G(\ell+1)}{G(k+\ell+1)} k\ell \left(\frac{1}{k}\frac{\partial A_{E,k,\ell}}{\partial z_1}(0,0) - \frac{1}{\ell}\frac{\partial A_{E,k,\ell}}{\partial z_2}(0,0)\right) \\ & \times \frac{2^{k\ell-1}}{D} |\Gamma(k\ell, -\log D)|. \end{aligned} \tag{39}$$

We remark that we have replaced the polynomial $\langle \log^h d\rangle_D$ by the incomplete Γ-function, and this will be explained in Section 8.

If, in addition to the conditions $k, \ell \in \mathbb{C}$, $\operatorname{Re}(k), \operatorname{Re}(\ell) >$, and $\operatorname{Re}(k+\ell) > -1$, we also have $k\ell \notin \mathbb{Z}_{<0}$, then we can extend $g_{k,\ell}$

$$g_{k,\ell} := \Gamma(k\ell+1)\frac{G(k+1)G(\ell+1)}{G(k+\ell+1)}.$$

This allows us to write the previous formula as

$$\begin{aligned} \langle \Upsilon_{k,\ell}(2\log d)\rangle_D \sim & g_{k,\ell} A_{E,k,\ell}(0,0)\frac{2^{k\ell}}{D}\frac{|\Gamma(k\ell+1, -\log D)|}{\Gamma(k\ell+1)} \\ & + g_{k,\ell}\left(\gamma(k+\ell-2) + \log\left(\frac{N_E}{4\pi^2}\right)\right) A_{E,k,\ell}(0,0)\frac{2^{k\ell-1}}{D}\frac{|\Gamma(k\ell, -\log D)|}{\Gamma(k\ell)} \\ & + g_{k,\ell}\left(\frac{1}{k}\frac{\partial A_{E,k,\ell}}{\partial z_1}(0,0) - \frac{1}{\ell}\frac{\partial A_{E,k,\ell}}{\partial z_2}(0,0)\right)\frac{2^{k\ell-1}}{D}\frac{|\Gamma(k\ell, -\log D)|}{\Gamma(k\ell)}. \end{aligned} \tag{40}$$

When $k\ell \in \mathbb{Z}_{<0}$, $g_{k,\ell}$ is not defined, but we can still evaluate (39). Thus, the condition $k\ell \notin \mathbb{Z}_{<0}$ is inessential.

We close this section by computing the derivatives of $A_{E,k,\ell}(z_1, z_2)$ by logarithmic differentiation. The derivatives will also be needed for numerical computations in the next section.

$$\frac{\frac{\partial A_{E,k,\ell}}{\partial z_1}(0,0)}{A_{E,k,\ell}(0,0)} = \sum_{p\nmid N_E} \frac{k\ell \log p}{p-1} + kp^{-1/2}\log p$$

$$\times \frac{(2p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell+1} + \xi_3(2\xi_3 p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k+1}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \xi_3^2(2\xi_3^2 p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k+1}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}}{\frac{3\delta_p}{1-\delta_p} + \left(\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell} + \mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}\right)}$$

$$+ \sum_{p\mid N_E} \frac{k\ell\log p}{p-1} - ka_p p^{-1/2}\log p \frac{\left(\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell+1} + \xi_3\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k+1}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \xi_3^2\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k+1}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}\right)}{\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell} + \mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}}$$

and

$$\frac{\frac{\partial A_{E,k,\ell}}{\partial z_2}(0,0)}{A_{E,k,\ell}(0,0)} = \sum_{p\nmid N_E} \frac{k\ell \log p}{1-p} - \ell p^{-1/2}\log p$$

$$\times \frac{(2p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell+1} + \xi_3(2\xi_3 p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell+1}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k} + \xi_3^2(2\xi_3^2 p^{-1/2}-a_p)\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell+1}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}}{\frac{3\delta_p}{1-\delta_p} + \left(\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell} + \mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}\right)}$$

$$+ \sum_{p\mid N_E} \frac{k\ell\log p}{1-p} + \ell a_p p^{-1/2}\log p \frac{\left(\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell+1} + \xi_3\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell+1}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k} + \xi_3^2\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell+1}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}\right)}{\mathcal{L}_p\left(\frac{1}{p^{1/2}}\right)^{k+\ell} + \mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{\ell} + \mathcal{L}_p\left(\frac{\xi_3^2}{p^{1/2}}\right)^{k}\mathcal{L}_p\left(\frac{\xi_3}{p^{1/2}}\right)^{\ell}}.$$

8. Numerical experiments

In this section we conduct some experiments comparing numerical formulas for the k,ℓ-moment $\left\langle L(1/2,E,\chi)^k L(1/2,E,\overline{\chi})^\ell\right\rangle_D$, with the approximation predicted by our model using equations (36) (when k,ℓ are nonnegative integers) and more generally (39) under the conditions

$$k,\ell\in\mathbb{C}, \mathrm{Re}(k), \mathrm{Re}(\ell), \text{and}\, \mathrm{Re}(k+\ell) > -1. \tag{41}$$

We continue to use the convention that $\mathrm{Re}(k)\geq \mathrm{Re}(\ell)$.

The experiments are done for the elliptic curves 11a1 and 14a1 and for $D = 3\cdot 10^6$. The amount of data that can be obtained for cubic twists is unfortunately limited compared to the case of quadratic twists, where one can use powerful results of Waldspurger [Wal80] and Kohnen–Zagier [KZ81] which relate the values of $L(1/2,E,\chi)$ to the Fourier coefficients of a weight 3/2 modular form. For the case of cubic twists, one has to rely on using the approximate functional equation to compute the value of $L(1/2,E,\chi)$.

To compute $L(1/2,E,\chi_d)$ for a primitive *quadratic* Dirichlet character χ_d, where d is a fundamental discriminant, Mao, Rodriguez-Villegas, and Tornaría [MRVT07] used generalized theta series associated to positive definite ternary quadratic forms. With their method, the computational complexity to compute the values $L(1/2,E,\chi_d)$ for $|d|\leq D$ is $O(D^{3/2})$.

To compute $L(1/2, E, \chi)$ for a primitive *cubic* Dirichlet character of conductor d, we use the approximate functional equation to write the critical L-value as

$$L(1/2, E, \chi) = \frac{2\pi}{d\sqrt{N_E}} \sum_{n=1}^{\infty} (\chi(n) + \varepsilon(E, \chi)\overline{\chi}(n)) \frac{a_n}{n} \exp\left(-\frac{2\pi n}{d\sqrt{N_E}}\right). \tag{42}$$

To obtain $L(1/2, E, \chi)$ with a desired precision, we need to take roughly $O(d)$ terms in the sum of (42). Thus, with this method, the computational complexity to compute the values $L(1/2, E, \chi)$ for $\chi \in \mathcal{F}_E(D)$ is $O(D^2)$.

The numerical values of $L(1/2, E, \chi)$ for 11a1 and 14a1 for $2 \cdot 10^6 \leq D \leq 3 \cdot 10^6$ were obtained by using 40 threads in the cluster of the Centre de Recherches Mathématiques (CRM) for a couple of months. The codes to evaluate $L(1/2, E, \chi)$ were created by using Cython built in SageMath [The12]. Moreover, those L-values up to $D = 2 \cdot 10^6$ were already computed by Jack Fearnley for the paper [DFK04], and we used his L-values. At least 9 decimal place accuracy is maintained in the numerical moments.

Finally, when doing numerical tests for a case when $k + \ell < 0$ or when there is a purely imaginary exponent involved, we use the nonvanishing subfamily $\mathcal{F}'_E(D)$ instead of $\mathcal{F}_E(D)$ defined as

$$\mathcal{F}'_E(D) := \{\chi \in \mathcal{F}_E(D) \mid L(1/2, E, \chi) \neq 0\}$$

on computing $\left\langle L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell \right\rangle_D$. For $D = 3 \cdot 10^6$, we have the following numbers of twists for $\mathcal{F}_E(D)$ and $\mathcal{F}'_E(D)$:

E	$\#\mathcal{F}_E(D)$	$\#\mathcal{F}'_E(D)$
11a1	778150	775686
14a1	605256	597822

Table 0. $\#\mathcal{F}_E(D)$ and $\#\mathcal{F}'_E(D)$ for 11a1 and 14a1 for $D = 3 \cdot 10^6$.

Regarding equation (36), we recall that k, ℓ are nonnegative integers and if $\ell = 0$, then only the first term is considered. We also evaluate moments in cases where k, ℓ are not integers but still satisfy conditions (41). For this, we evaluate equation (39). The polynomial $(k\ell)! \sum_{j=0}^{k\ell} \frac{(-1)^j \log^{k\ell - j} D}{(k\ell - j)!}$ is replaced by the incomplete Gamma function which is well implemented in standard mathematical software. The incomplete Gamma function is defined by

$$\Gamma(a, z) = \int_z^\infty t^{h-1} e^{-t} dt.$$

$\Gamma(a, z)$ satisfies for $a \in \mathbb{C}$ and n fixed

$$\Gamma(a, z) = z^{a-1} e^{-z} \left(\sum_{k=0}^{n-1} \frac{a(a-1)\cdots(a-k+1)}{z^k} + O_a(z^{-n}) \right), \text{ as } z \to \infty, |\arg z| < \frac{3\pi}{2}.$$

Moreover, the formula is exact for $a \in \mathbb{Z}_{>0}$ (see formulas 6.5.32 and 6.5.13 in [AS64]). By setting $a = h + 1$ and $z = -\log D$ above and combining with Corollary 3.2, we conclude the following.

Remark 8.1. For $h \in \mathbb{R}$,

$$\left\langle \log^h d \right\rangle_D \sim \frac{1}{D} |\Gamma(h + 1, -\log D)|.$$

8.1. **Results for k, ℓ nonnegative integers.** We have computed the moments for $(k,\ell) = (1,0),(2,0),(1,1)$, and $(2,1)$. In the cases of $(k,\ell) = (1,0),(2,0)$ our conjecture predicts that $P_{k\ell}(x)$ is a constant. When $(k,\ell) = (1,1)$, a polynomial of degree 1 is predicted, and for the case of $(k,\ell) = (2,1)$, we expect a polynomial of degree 2. Since we have only considered the first two coefficients, our computation of $P_{k\ell}(x)$ is missing the constant coefficient for $(2,1)$. Nevertheless, the approximation is still good. Our results are recorded in Tables 1 and 2. We present two ways of predicting the moment. The first one, indicated as "$\log^{k\ell}$", is calculated with

$$g_{k,\ell} A_{E,k,\ell}(0,0) 2^{k\ell} \frac{\log^{k\ell} D}{(k\ell)!},$$

while the one denoted by "$\log^{k\ell} + \log^{k\ell-1}$" is calculated from equation (36). Of course, when $k\ell = 0$, the conjecture simply predicts the constant $a_{k\ell}$. The column "moment" indicates the numerical moment. The column "quotient" indicates the quotient between the numerical moment and the one predicted in column "$\log^{k\ell} + \log^{k\ell-1}$". We see in general that using two terms is a better approximation than simply using the main term.

In Figures 1 and 2, the quotient is plotted as a function of all values of D up to $3 \cdot 10^6$ sampled at intervals of size 10^4. Note that the outliers (which all lie in $D \leq 5 \cdot 10^5$) are removed in those figures. We see that the convergence is relatively slow compared to the usual cases involving quadratic twists.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(1,0)$	1	0.9369	0.9369	0.9369	0.9410	1.0044
$(2,0)$	1	-1.9659	-1.9659	-1.9659	-1.9648	0.9994
$(1,1)$	1	1.6516	24.632	24.015	24.100	1.0035
$(2,1)$	1	0.8744	97.372	111.96	111.97	1.0001

TABLE 1. Integral moments for 11a1 with $D = 3 \cdot 10^6$.

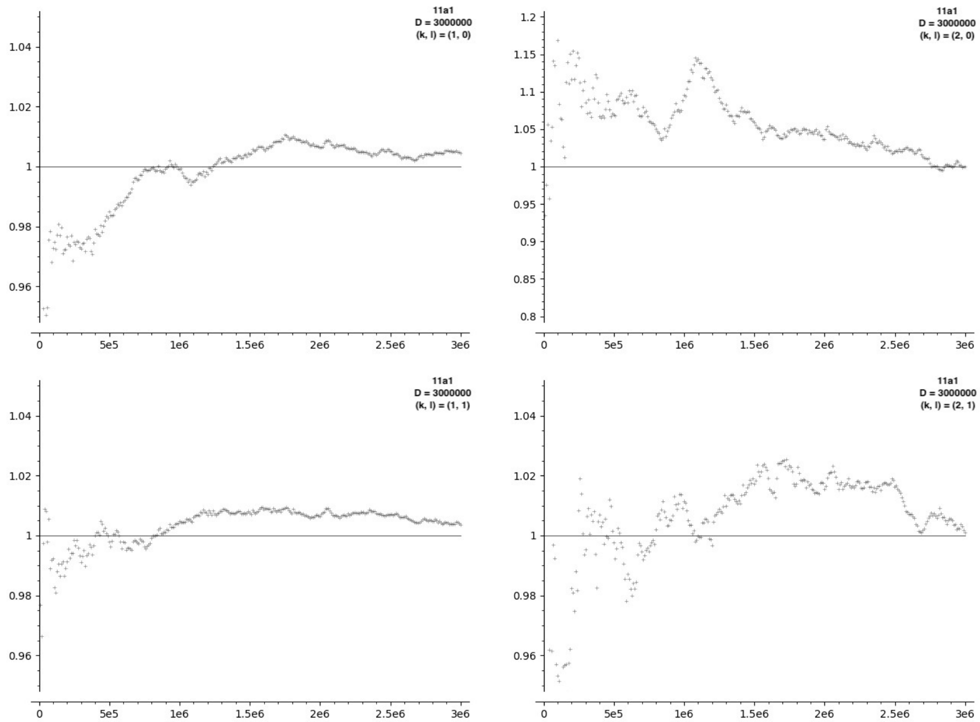

FIGURE 1. Convergence of integral moments for 11a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(1,0)$	1	1.0394	1.0394	1.0394	1.0369	0.9976
$(2,0)$	1	0.6467	0.6467	0.6467	0.6247	0.9660
$(1,1)$	1	0.8899	13.277	13.188	13.240	1.0039
$(2,1)$	1	0.4987	55.540	65.580	65.344	0.9964

TABLE 2. Integral moments for 14a1 with $D = 3 \cdot 10^6$.

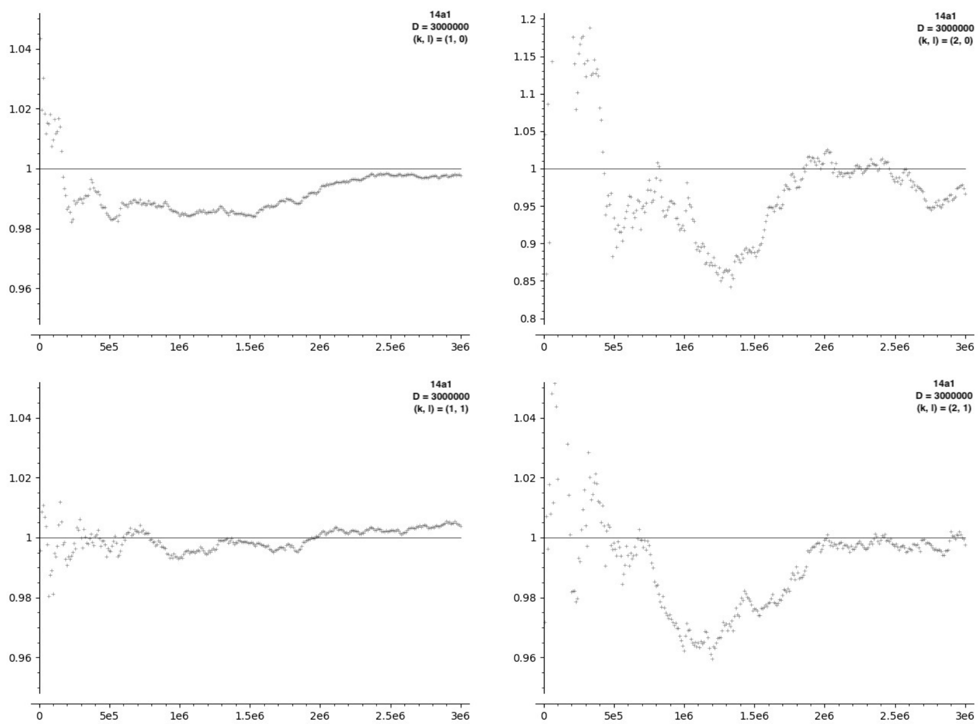

FIGURE 2. Convergence of integral moments for 14a1 with $D \leq 3 \cdot 10^6$

8.2. **Results for other real values of k, ℓ.** The numerical evaluation of non-integral moments presents some computational challenges due to having to specify a choice for the branch of the logarithm. For the moments of the Riemann zeta function, $\log \zeta(1/2+it)$ and then $\arg \zeta(1/2+it)$ are defined by continuous variation of the imaginary part t, and similarly in the work of Keating and Snaith for the moments of $Z(U,\theta)$ where $\arg Z(U,\theta)$ is defined by continuous variation of θ, and (6) indicates a clear choice for the arguments defining the moments. In our case, we are dealing with moments of a discrete family (there is no continuous variation of the conductor). When doing numerical testing, the value of $L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell$ is obtained with the principal branch of the logarithm, i.e. with argument in $(-\pi, \pi]$. Furthermore, the conjectural prediction for the moments involves the Euler product (35), where the numerical computations are performed at the level of each factor. We then have to restrict our computations to cases where both computations lead the same results.

To see the influence of both $L(1/2, E, \chi)^k L(1/2, E, \overline{\chi})^\ell$ and the Euler product, let $L = L_1 e^{i\delta}$ be a nonzero complex number with $L_1 = |L|$ and $-\pi \leq \delta < \pi$ such that $L = AB$. Similarly write $A = A_1 e^{i\alpha}$ and $B = B_1 e^{i\alpha}$ with $A_1 = |A|, B_1 = |B|$ and $-\pi \leq \alpha, \beta < \pi$. On the one hand, equation (6) indicates that we should consider

$$L^k \overline{L}^\ell = L^{k+\ell} e^{i(k-\ell)\delta} = L^{k+\ell} e^{i[(k-\ell)\delta]_{[-\pi,\pi)}}, \tag{43}$$

where $[x]_{[-\pi,\pi)}$ indicates the representative of x in the interval $[-\pi, \pi)$, namely, the unique $x + 2\pi n \in [-\pi, \pi)$ for $n \in \mathbb{Z}$.

On the other hand,

$$(A^k \overline{A}^\ell)(B^k \overline{B}^\ell) = (AB)^{k+\ell} e^{i[(k-\ell)\alpha]_{[-\pi,\pi)}} e^{i[(k-\ell)\beta]_{[-\pi,\pi)}}. \tag{44}$$

However, we have that

$$[(k-\ell)\delta]_{[-\pi,\pi)} = [(k-\ell)[\alpha+\beta]_{[-\pi,\pi)}]_{[-\pi,\pi)} \text{ and } [(k-\ell)\alpha]_{[-\pi,\pi)} + [(k-\ell)\beta]_{[-\pi,\pi)}$$

do not always have the same representative in $[-\pi,\pi)$ when k,ℓ are not integers.

The conjecture requires that all the computations be performed as in equation (43), but the numerical computations of the product $L(1/2,E,\chi)^k L(1/2,E,\overline{\chi})^\ell$ and of the Euler product (35) are performed at the level of each factor, resulting in computations as in equation (44).

A way to avoid this issue is to explore cases of (k,ℓ) where both computations yield the same result when performed by standard software. This could happen in cases in which (43) and (44) yield always the same result (such as $k=\ell$, $k=\ell+1$), or in cases in which the discrepancy is very small and there is a clear default in how the software computes the arguments because the factors in the Euler product are close to 1 (for example, when $\ell=0$ with k small, or $k+\ell=0$, $k+\ell=1$ with $|\ell|$ small).

We have computed the moments for k,ℓ real nonnegative for $(k,\ell)=(1/n,0)$ for $n=2,\dots,10$. As before, in this case the model predicts a constant polynomial. Our results are recorded in Tables 3 and 4. The last column of each table, denoted by "quotient", indicates the quotient of the numerical value for the moment and the prediction $a_{k,\ell}$.

Figures 3 and 4 illustrate the convergence for the values $(1/n,0)$. The values seem to be quite regular and stable. However, we notice that they do not seem to approach 1 as n goes to infinity, specially in the case of 14a1. We speculate that this is due to insufficient data, namely, that our D is too small.

(k,ℓ)	**11a1**		
	$a_{k\ell}$	moment	quotient
$(1/2,0)$	1.0924352	1.1064131	1.012795
$(1/3,0)$	1.0736431	1.0809212	1.006779
$(1/4,0)$	1.0579795	1.0619068	1.003712
$(1/5,0)$	1.0472890	1.0493806	1.001997
$(1/6,0)$	1.0397736	1.0407432	1.000932
$(1/7,0)$	1.0342601	1.0344846	1.000217
$(1/8,0)$	1.0300620	1.0297604	0.999707
$(1/9,0)$	1.0267666	1.0260758	0.999327
$(1/10,0)$	1.0241146	1.0231252	0.999034

Table 3. Rational moments with a positive and a zero parameter for 11a1 with $D=3\cdot 10^6$.

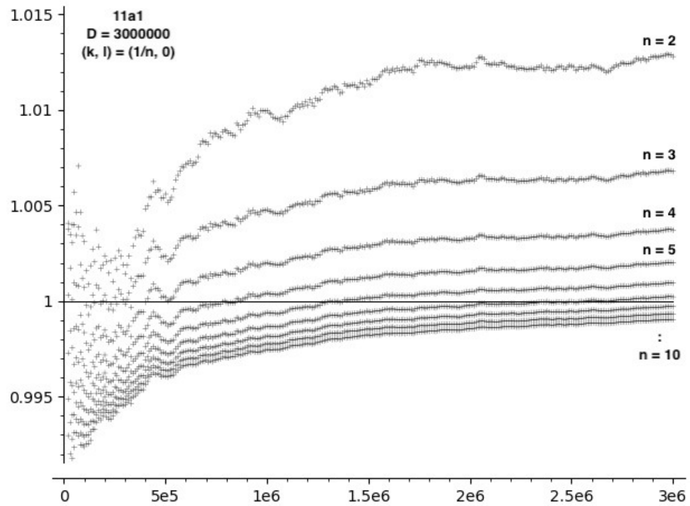

FIGURE 3. Rational moments with a positive and a zero parameter for 11a1 with $D \leq 3 \cdot 10^6$.

(k, ℓ)	**14a1**		
	$a_{k\ell}$	moment	quotient
$(1/2, 0)$	1.0450753	1.0499353	1.0046504
$(1/3, 0)$	1.0323368	1.0333675	1.0009984
$(1/4, 0)$	1.0246570	1.0230284	0.9984105
$(1/5, 0)$	1.0198069	1.0163783	0.9966380
$(1/6, 0)$	1.0165123	1.0118002	0.9953645
$(1/7, 0)$	1.0141414	1.0084705	0.9944081
$(1/8, 0)$	1.0123583	1.0059443	0.9936643
$(1/9, 0)$	1.0109705	1.0039639	0.9930694
$(1/10, 0)$	1.0098607	1.0023704	0.9925828

TABLE 4. Rational moments with a positive and a zero parameter for 14a1 with $D = 3 \cdot 10^6$.

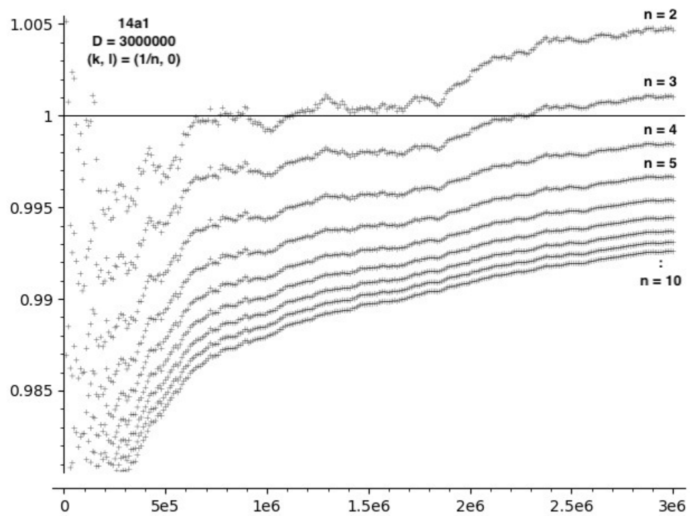

FIGURE 4. Rational moments with a positive and a zero parameter for 14a1 with $D \leq 3 \cdot 10^6$.

We have also computed the moments of the form $(1/n, 1/n)$ for $n = 2, \ldots, 10$, as well as $(3/2, 3/2)$. As discussed earlier, these moments correspond to powers of the absolute value $|L(1/2, E, \chi)|$. The results are recorded in Tables 5 and 6. As before, the column "$\log^{k\ell}$" denotes

$$g_{k,\ell} A_{E,k,\ell}(0,0) 2^{k\ell} \frac{\log^{k\ell} D}{\Gamma(k\ell+1)},$$

while the column "$\log^{k\ell} + \log^{k\ell-1}$" is calculated from equation (40). Again, using the first two terms (together with the incomplete Gamma function) is a better approximation than simply using the first term.

Figures 5 and 6 illustrate the convergence. In these graphs and all the subsequent figures in the article, the quotient has been computed using the first coefficient, namely, the value from the column "$\log^{k\ell}$". This is a slightly less good approximation than using the computation of column "$\log^{k\ell} + \log^{k\ell-1}$", but it is sufficient to see the convergence towards the value $D = 3 \cdot 10^6$ indicated in the corresponding tables. We see that the data seems to approach 1, and that the data is still very unstable for the moment $(3/2, 3/2)$.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(3/2, 3/2)$	1.1444	1.3693	268.69	313.98	317.49	1.0112
$(1/2, 1/2)$	1.0362	1.3685	3.0748	2.9792	2.9852	1.0020
$(1/3, 1/3)$	1.0341	1.2266	1.8086	1.7778	1.7787	1.0005
$(1/4, 1/4)$	1.0264	1.1596	1.4567	1.4415	1.4415	1.0000
$(1/5, 1/5)$	1.0203	1.1222	1.3038	1.2947	1.2943	0.9997
$(1/6, 1/6)$	1.0158	1.0985	1.2216	1.2156	1.2149	0.9994
$(1/7, 1/7)$	1.0126	1.0823	1.1715	1.1671	1.1664	0.9994
$(1/8, 1/8)$	1.0103	1.0706	1.1383	1.1350	1.1341	0.9992
$(1/9, 1/9)$	1.0085	1.0618	1.1149	1.1124	1.1113	0.9990
$(1/10, 1/10)$	1.0071	1.0549	1.0978	1.0957	1.0945	0.9990

TABLE 5. Positive moments of $|L(1/2, E, \chi)|$ for 11a1 with $D = 3 \cdot 10^6$.

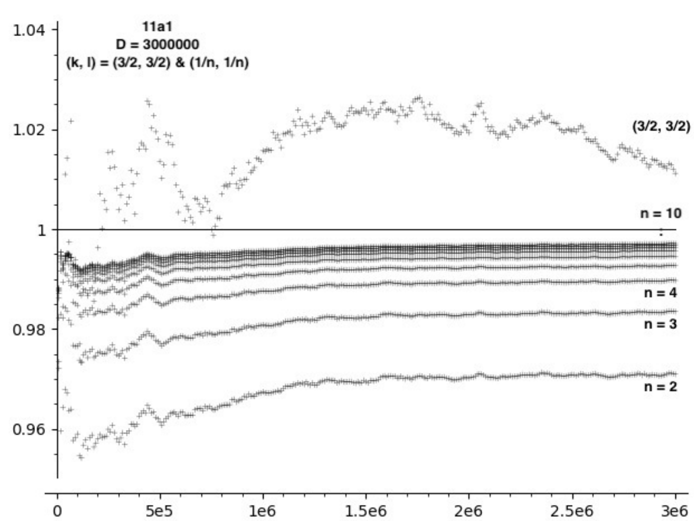

FIGURE 5. Positive moments of $|L(1/2, E, \chi)|$ for 11a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(3/2,3/2)$	1.1444	0.5207	102.19	122.76	124.97	1.0180
$(1/2,1/2)$	1.0362	1.0526	2.3650	2.3046	2.3116	1.0028
$(1/3,1/3)$	1.0341	1.0504	1.5487	1.5262	1.5274	1.0008
$(1/4,1/4)$	1.0264	1.0419	1.3087	1.2969	1.2962	0.9994
$(1/5,1/5)$	1.0203	1.0351	1.2026	1.1953	1.1934	0.9984
$(1/6,1/6)$	1.0158	1.0300	1.1454	1.1404	1.1375	0.9975
$(1/7,1/7)$	1.0126	1.0261	1.1106	1.1070	1.1034	0.9967
$(1/8,1/8)$	1.0103	1.0230	1.0877	1.0849	1.0807	0.9961
$(1/9,1/9)$	1.0085	1.0206	1.0717	1.0696	1.0648	0.9955
$(1/10,1/10)$	1.0071	1.0187	1.0601	1.0583	1.0530	0.9950

TABLE 6. Positive moments of $|L(1/2,E,\chi)|$ for 14a1 with $D=3\cdot 10^6$.

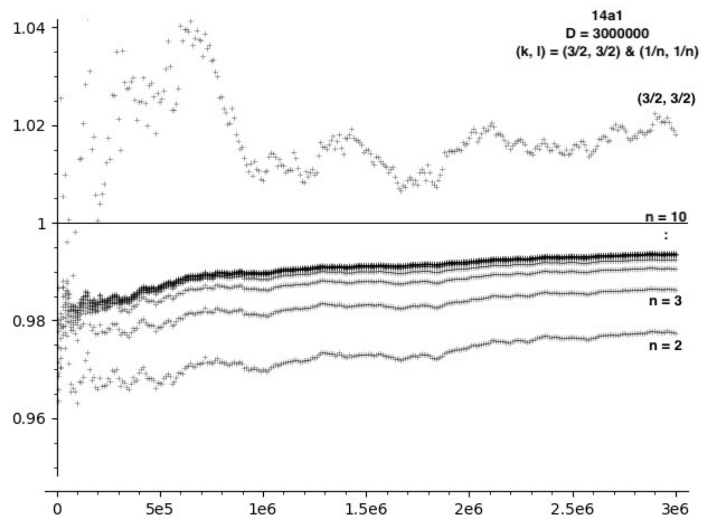

FIGURE 6. Positive moments of $|L(1/2,E,\chi)|$ for 14a1 with $d\le 3\cdot 10^6 6$.

We have similarly computed the moments of the form $(1+1/n,1/n)$ for $n=1,\dots,10$. These moments correspond to powers of $|L(1/2,E,\chi)|$ combined with powers of $L(1/2,E,\chi)$. The results are recorded in Tables 7 and 8.

Figures 7 and 8 illustrate the convergence. We see that the data is quite unstable, but it still could approach 1 with some delay.

(k, ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(3/2, 1/2)$	0.9312	1.0634	8.1802	8.0262	8.0627	1.0045
$(4/3, 1/3)$	0.9568	1.0444	3.7503	3.6524	3.6745	1.0061
$(5/4, 1/4)$	0.9719	1.0234	2.5839	2.5233	2.5385	1.0060
$(6/5, 1/5)$	0.9804	1.0081	2.0810	2.0387	2.0506	1.0058
$(7/6, 1/6)$	0.9857	0.9970	1.8074	1.7754	1.7854	1.0057
$(8/7, 1/7)$	0.9891	0.9888	1.6371	1.6116	1.6205	1.0055
$(9/8, 1/8)$	0.9914	0.9825	1.5216	1.5005	1.5085	1.0054
$(10/9, 1/9)$	0.9931	0.9776	1.4383	1.4203	1.4278	1.0053
$(11/10, 1/10)$	0.9943	0.9736	1.3755	1.3599	1.3669	1.0052

TABLE 7. Positive moments of the form $L(1/2, E, \chi)|L(1/2, E, \chi)|^\ell$ for 11a1 with $D = 3 \cdot 10^6$.

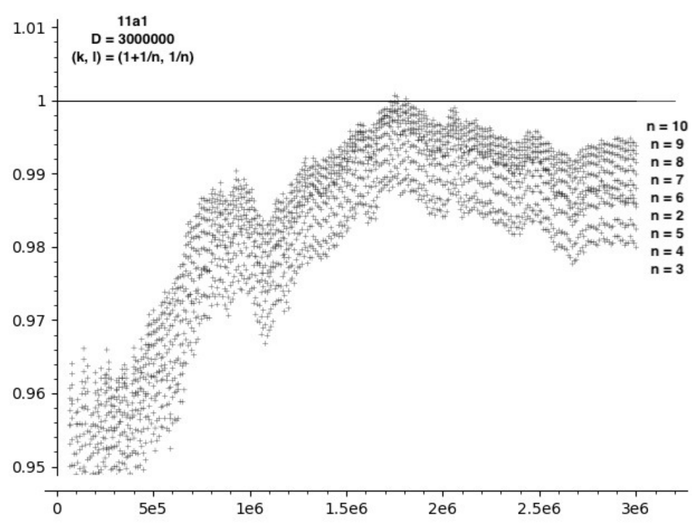

FIGURE 7. Positive moments of the form $L(1/2, E, \chi)|L(1/2, E, \chi)|^\ell$ for 11a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(3/2,1/2)$	0.9312	0.8596	6.6120	6.5848	6.5569	0.99576
$(4/3,1/3)$	0.9568	0.9465	3.3987	3.3410	3.3340	0.99790
$(5/4,1/4)$	0.9719	0.9800	2.4743	2.4322	2.4281	0.99830
$(6/5,1/5)$	0.9804	0.9967	2.0575	2.0259	2.0224	0.99829
$(7/6,1/6)$	0.9857	1.0064	1.8244	1.7994	1.7963	0.99828
$(8/7,1/7)$	0.9891	1.0127	1.6767	1.6562	1.6532	0.99820
$(9/8,1/8)$	0.9914	1.0171	1.5751	1.5578	1.5549	0.99813
$(10/9,1/9)$	0.9931	1.0203	1.5011	1.4861	1.4833	0.99813
$(11/10,1/10)$	0.9943	1.0227	1.4449	1.4317	1.4290	0.99808

TABLE 8. Positive moments of the form $L(1/2,E,\chi)|L(1/2,E,\chi)|^{\ell}$ for 14a1 with $D=3\cdot 10^6$.

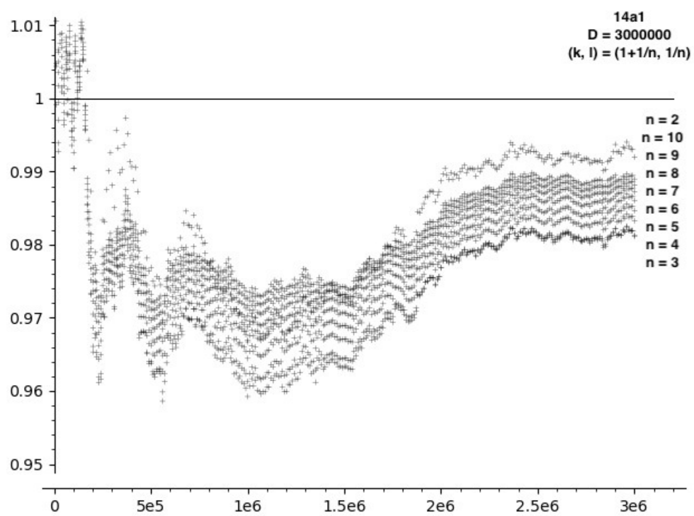

FIGURE 8. Positive moments of the form $L(1/2,E,\chi)|L(1/2,E,\chi)|^{\ell}$ for 14a1 with $D\le 3\cdot 10^6$.

In addition we have considered several fractional exponents satisfying $k>0>\ell$. This was done by removing the terms $L(1/2,E,\chi)=0$. As explained after equation (38), this was not an issue if we chose $k+\ell\ge 0$ and those terms could be safely ignored. The (k,ℓ) were again chosen in such a way that equations (43) and (44) yield the same numbers. We obtained good matches for large n, as illustrated in Tables 9 and 10, and Figures 9 and 10, corresponding to $(1/n,-1/n)$, and 11 and 12 and Figures 11 and 12, corresponding to $(1+1/n,-1/n)$.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(1/2,-1/2)$	0.7904	0.8322	0.2732	0.2860	0.2871	1.0039
$(1/3,-1/3)$	0.8973	0.9316	0.5744	0.5870	0.6021	1.0258
$(1/4,-1/4)$	0.9402	0.9627	0.7349	0.7441	0.7565	1.0166
$(1/5,-1/5)$	0.9611	0.9765	0.8220	0.8286	0.8372	1.0104
$(1/6,-1/6)$	0.9728	0.9838	0.8731	0.8780	0.8837	1.0065
$(1/7,-1/7)$	0.9799	0.9882	0.9052	0.9090	0.9127	1.0041
$(1/8,-1/8)$	0.9845	0.9909	0.9267	0.9296	0.9319	1.0025
$(1/9,-1/9)$	0.9878	0.9929	0.9417	0.9440	0.9453	1.0014
$(1/10,-1/10)$	0.9900	0.9942	0.9525	0.9544	0.9549	1.0005

TABLE 9. Positive moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 11a1 with $D = 3 \cdot 10^6$.

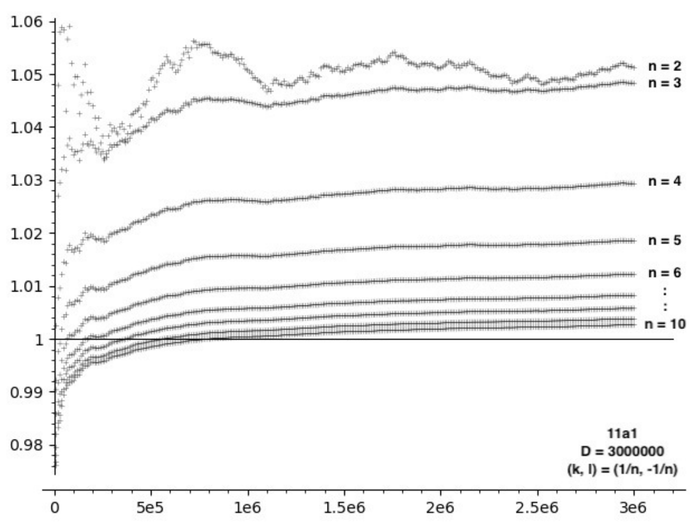

FIGURE 9. Positive moments of the form $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 11a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(1/2,-1/2)$	0.7904	1.0333	0.3391	0.3543	0.3549	1.0017
$(1/3,-1/3)$	0.8973	1.0159	0.6264	0.6389	0.6430	1.0065
$(1/4,-1/4)$	0.9402	1.0091	0.7704	0.7791	0.7793	1.0002
$(1/5,-1/5)$	0.9611	1.0059	0.8468	0.8529	0.8497	0.9963
$(1/6,-1/6)$	0.9728	1.0041	0.8911	0.8956	0.8901	0.9939
$(1/7,-1/7)$	0.9799	1.0030	0.9189	0.9223	0.9152	0.9923
$(1/8,-1/8)$	0.9845	1.0023	0.9373	0.9400	0.9318	0.9913
$(1/9,-1/9)$	0.9878	1.0018	0.9502	0.9523	0.9433	0.9906
$(1/10,-1/10)$	0.9900	1.0015	0.9594	0.9612	0.9516	0.9900

TABLE 10. Positive moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 14a1 with $D = 3 \cdot 10^6$.

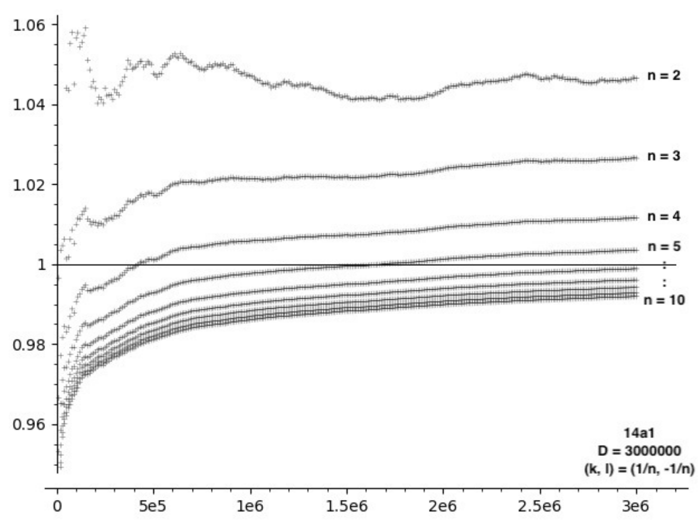

FIGURE 10. Positive moments of the form $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 14a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(3/2,-1/2)$	2.0725	-1.2263	-0.0923	-0.1042	-0.1044	1.0019
$(4/3,-1/3)$	1.1901	-0.1717	-0.0384	-0.0430	-0.0866	2.0140
$(5/4,-1/4)$	1.0801	0.2071	0.0729	0.0740	0.0277	0.3743
$(6/5,-1/5)$	1.0440	0.3968	0.1786	0.1826	0.1417	0.7760
$(7/6,-1/6)$	1.0278	0.5095	0.2673	0.2728	0.2371	0.8691
$(8/7,-1/7)$	1.0192	0.5838	0.3400	0.3461	0.3153	0.9110
$(9/8,-1/8)$	1.0139	0.6364	0.3997	0.4060	0.3793	0.9342
$(10/9,-1/9)$	1.0106	0.6754	0.4492	0.4555	0.4321	0.9486
$(11/10,-1/10)$	1.0083	0.7056	0.4907	0.4969	0.4763	0.9585

TABLE 11. Positive moments of the form $L(1/2,E,\chi)\left(\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}\right)^{\ell}$ for 11a1 with $D = 3\cdot 10^6$.

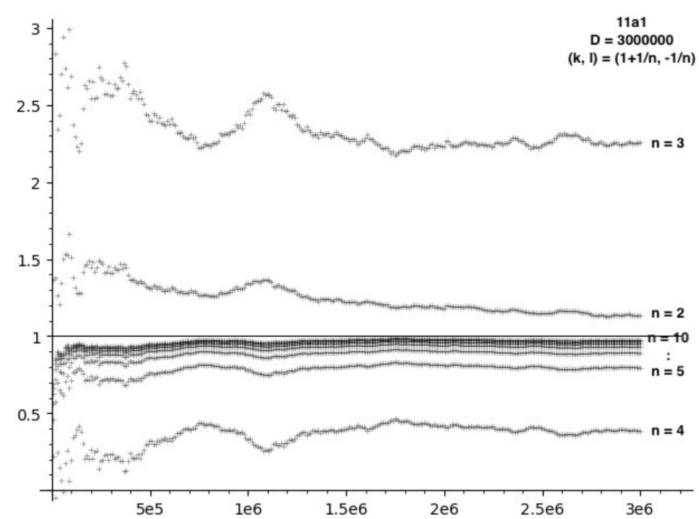

FIGURE 11. Positive moments of the form $L(1/2,E,\chi)\left(\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}\right)^{\ell}$ for 11a1 with $D \leq 3\cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(3/2,-1/2)$	2.0725	0.8750	0.0658	0.0705	0.0721	1.0231
$(4/3,-1/3)$	1.1901	0.9663	0.2161	0.2255	0.2074	0.9200
$(5/4,-1/4)$	1.0801	0.9951	0.3503	0.3612	0.3406	0.9430
$(6/5,-1/5)$	1.0440	1.0084	0.4539	0.4648	0.4454	0.9582
$(7/6,-1/6)$	1.0278	1.0157	0.5330	0.5433	0.5257	0.9676
$(8/7,-1/7)$	1.0192	1.0204	0.5943	0.6040	0.5881	0.9737
$(9/8,-1/8)$	1.0139	1.0235	0.6429	0.6520	0.6375	0.9778
$(10/9,-1/9)$	1.0106	1.0258	0.6823	0.6907	0.6774	0.9808
$(11/10,-1/10)$	1.0083	1.0275	0.7147	0.7226	0.7103	0.9830

TABLE 12. Positive moments of the form $L(1/2,E,\chi)\left(\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}\right)^{\ell}$ for 14a1 with $D=3\cdot 10^6$.

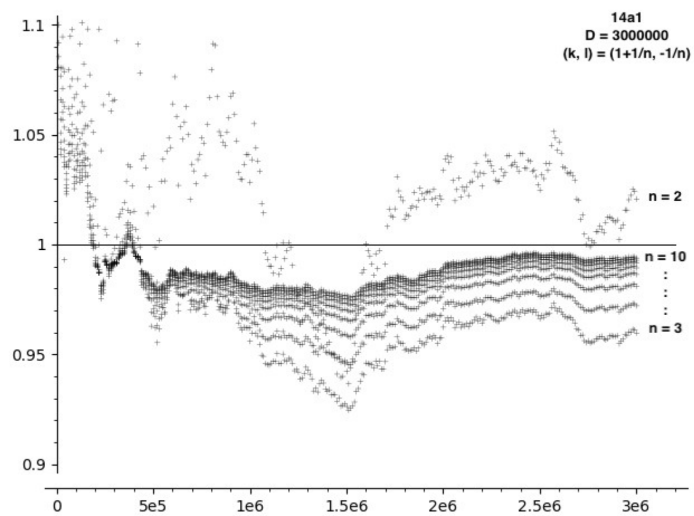

FIGURE 12. Positive moments of the form $L(1/2,E,\chi)\left(\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}\right)^{\ell}$ for 14a1 with $D\le 3\cdot 10^6$.

Finally, we have also considered some cases within the conditions (41) with $k\ge 0>\ell$ and $k+\ell<0$. These cases present difficulties as discussed after equation (38). For example, we must divide by $\#\mathcal{F}'_E(D)$ instead of $\#\mathcal{F}_E(D)$, since we must completely ignore the zero terms.

The values we obtained are very stable. We list them in Tables 13 and 14 and Figures 13 and 14, where $(0,-1/n)$ is considered, and Tables 15 and 16 and Figures 15 and 16, where $(-1/n,-1/n)$ is considered.

(k,ℓ)	**11a1**		
	$a_{k\ell}$	moment	quotient
$(0,-1/2)$	0.9158795	0.8954299	0.9776721
$(0,-1/3)$	0.9320485	0.9261987	0.9937238
$(0,-1/4)$	0.9456153	0.9421381	0.9963229
$(0,-1/5)$	0.9551282	0.9524591	0.9972055
$(0,-1/6)$	0.9619489	0.9597014	0.9976636
$(0,-1/7)$	0.9670249	0.9650543	0.9979622
$(0,-1/8)$	0.9709315	0.9691658	0.9981814
$(0,-1/9)$	0.9740236	0.9724195	0.9983531
$(0,-1/10)$	0.9765285	0.9750568	0.9984929

TABLE 13. Rational moments with a negative and a zero parameter for 11a1 with $D = 3\cdot 10^6$.

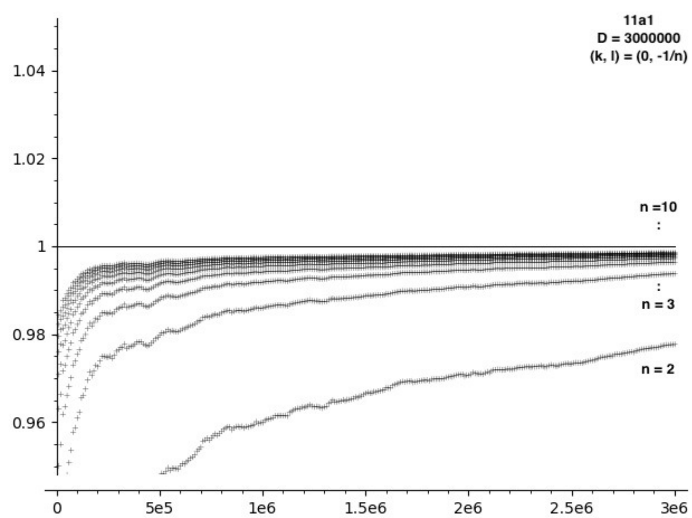

FIGURE 13. Rational moments with a negative and a zero parameter for 11a1 with $D \leq 3\cdot 10^6$.

(k,ℓ)	**14a1**, $D = 3 \cdot 10^6$		
	$a_{k\ell}$	moment	quotient
$(0,-1/2)$	0.96782787	0.90653233	0.9366669
$(0,-1/3)$	0.97368552	0.94499539	0.9705345
$(0,-1/4)$	0.97878866	0.96018439	0.9809926
$(0,-1/5)$	0.98241577	0.96863406	0.9859716
$(0,-1/6)$	0.98503776	0.97408115	0.9888770
$(0,-1/7)$	0.98700029	0.97790152	0.9907814
$(0,-1/8)$	0.98851726	0.98073453	0.9921269
$(0,-1/9)$	0.98972207	0.98292110	0.9931284
$(0,-1/10)$	0.99070076	0.98466066	0.9939032

TABLE 14. Rational moments with a negative and a zero parameter for 14a1 with $D = 3 \cdot 10^6$.

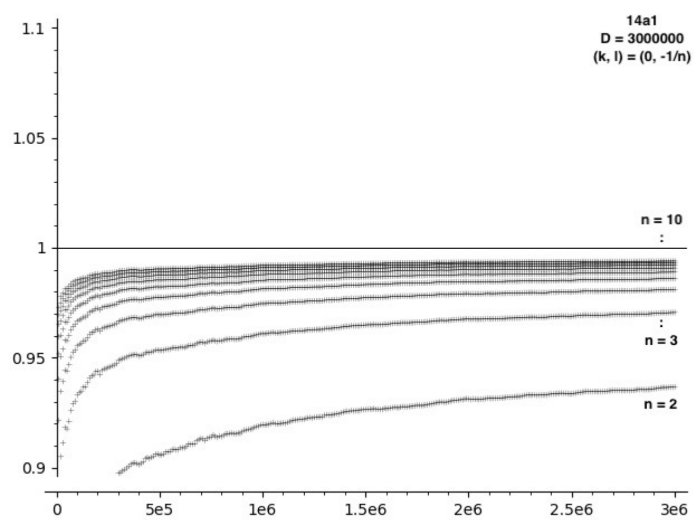

FIGURE 14. Rational moments with a negative and a zero parameter for 14a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(-1/3,-1/3)$	1.4284	0.9207	1.8752	1.8265	1.5295	0.8374
$(-1/4,-1/4)$	1.1553	0.9254	1.3082	1.2898	1.2121	0.9398
$(-1/5,-1/5)$	1.0805	0.9329	1.1479	1.1377	1.1015	0.9682
$(-1/6,-1/6)$	1.0492	0.9400	1.0796	1.0731	1.0514	0.9798
$(-1/7,-1/7)$	1.0331	0.9461	1.0447	1.0401	1.0251	0.9856
$(-1/8,-1/8)$	1.0238	0.9512	1.0248	1.0213	1.0102	0.9891
$(-1/9,-1/9)$	1.0179	0.9554	1.0126	1.0100	1.0011	0.9912
$(-1/10,-1/10)$	1.0139	0.9591	1.0048	1.0027	0.9954	0.9927

Table 15. Negative moments of $|L(1/2,E,\chi)|$ for 11a1 with $D=3\cdot 10^6$.

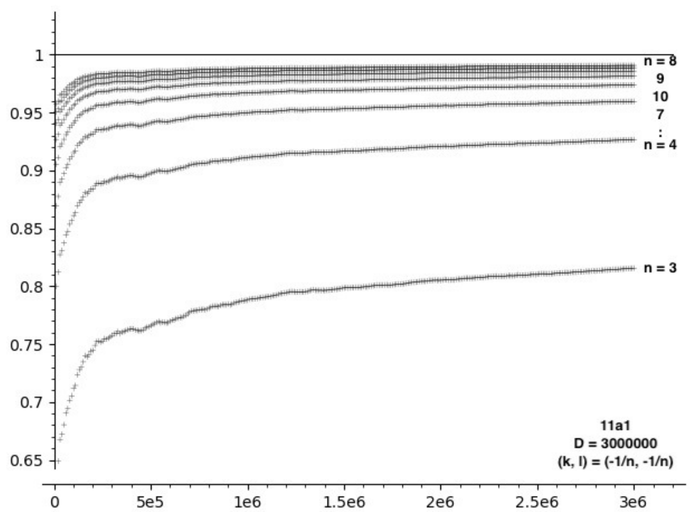

Figure 15. Negative moments of $|L(1/2,E,\chi)|$ for 11a1 with $D\leq 3\cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
$(-1/3,-1/3)$	1.4284	0.9371	1.9085	1.8619	1.4466	0.7770
$(-1/4,-1/4)$	1.1553	0.9520	1.3459	1.3282	1.1954	0.9000
$(-1/5,-1/5)$	1.0805	0.9613	1.1827	1.1731	1.1031	0.9403
$(-1/6,-1/6)$	1.0492	0.9676	1.1113	1.1051	1.0596	0.9588
$(-1/7,-1/7)$	1.0331	0.9722	1.0735	1.0692	1.0360	0.9690
$(-1/8,-1/8)$	1.0238	0.9756	1.0511	1.0479	1.0221	0.9753
$(-1/9,-1/9)$	1.0179	0.9783	1.0368	1.0344	1.0133	0.9796
$(-1/10,-1/10)$	1.0139	0.9805	1.0272	1.0252	1.0075	0.9827

Table 16. Negative moments of $|L(1/2,E,\chi)|$ for 14a1 with $D=3\cdot 10^6$.

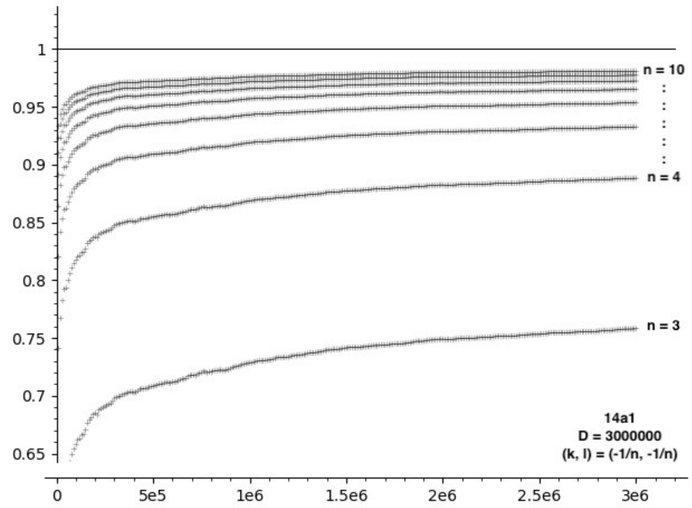

FIGURE 16. Negative moments of $|L(1/2,E,\chi)|$ for 14a1 with $D \le 3\cdot 10^6$.

8.3. **Results for complex values of k,ℓ.** We consider several purely imaginary moments. As in the case of $k+\ell<0$, we must divide by $\#\mathcal{F}'_E(D)$ instead of $\#\mathcal{F}_E(D)$ when computing the numerical moment, since we have to completely ignore the zero terms.

The first family that we consider is $(k,\ell)=(ni,0)$, for $n=1,2$ as well as $1/n=2,\ldots,10$. The results are listed in Tables 17 and 18 and Figures 17 and 18.

Then we consider $(i/n,-i/n)$ and $(i/n,i/n)$ for $n=1,\ldots,10$. The results are listed in Tables 19, 20, 21, and 22 and Figures 19, 20, 21, and 22. The case (i,i) satisfies that $k\ell=-1$ and therefore $g_{i,i}$ is not well-defined. We record the value of $g_{i,i}/\Gamma(0)$ in its place.

In all the tables, the word "quotient" indicates the absolute value of the quotient of the numerical moment and the moment predicted by the conjecture. As before, in the figures, the quotient is computed by using the first coefficient, namely, the value from the column "$\log^{k\ell}$". We see that these purely imaginary results are very stable.

(k,ℓ)	**11a1**, $D=3\cdot 10^6$		
	$a_{k\ell}$	moment	quotient
$(2i,0)$	$-0.1151+1.6647i$	$-1.1425+0.8930i$	0.8690
$(i,0)$	$0.9025+0.4665i$	$0.7676+0.4864i$	0.8944
$(i/2,0)$	$0.9877+0.1507i$	$0.9688+0.1568i$	0.9823
$(i/3,0)$	$0.9955+0.0893i$	$0.9891+0.0940i$	0.9941
$(i/4,0)$	$0.9976+0.0639i$	$0.9945+0.0679i$	0.9972
$(i/5,0)$	$0.9985+0.0500i$	$0.9967+0.0534i$	0.9984
$(i/6,0)$	$0.9990+0.0412i$	$0.9978+0.0441i$	0.9989
$(i/7,0)$	$0.9993+0.0350i$	$0.9984+0.0375i$	0.9992
$(i/8,0)$	$0.9994+0.0305i$	$0.9988+0.0327i$	0.9995
$(i/9,0)$	$0.9995+0.0270i$	$0.9991+0.0290i$	0.9996
$(i/10,0)$	$0.9996+0.0242i$	$0.9992+0.0261i$	0.9997

TABLE 17. Complex moment with a zero parameter for 11a1 with $D = 3 \cdot 10^6$.

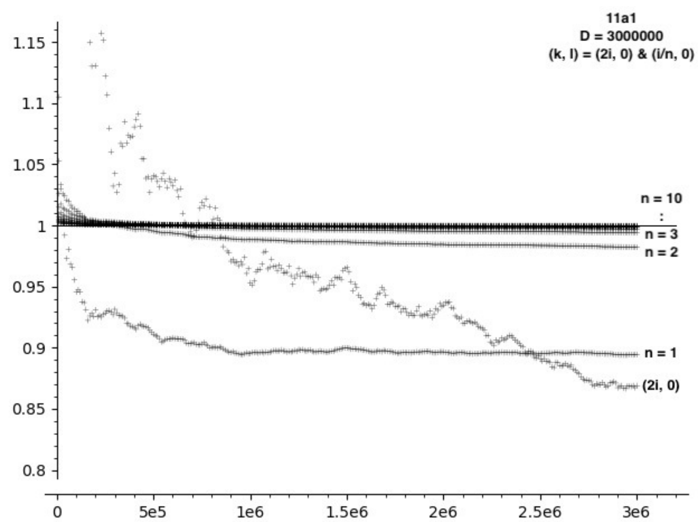

FIGURE 17. Complex moment with a zero parameter for 11a1 with $D \leq 3 \cdot 10^6$.

(k, ℓ)	**14a1**, $D = 3 \cdot 10^6$		
	$a_{k\ell}$	moment	quotient
$(2i, 0)$	$0.6458 + 0.8093i$	$0.2451 + 0.8760i$	0.8785
$(i, 0)$	$0.9606 + 0.1743i$	$0.9286 + 0.2159i$	0.9765
$(i/2, 0)$	$0.9923 + 0.0579i$	$0.9982 + 0.0782i$	1.0073
$(i/3, 0)$	$0.9967 + 0.0350i$	$1.0010 + 0.0507i$	1.0050
$(i/4, 0)$	$0.9982 + 0.0253i$	$1.0010 + 0.0378i$	1.0032
$(i/5, 0)$	$0.9988 + 0.0199i$	$1.0008 + 0.0302i$	1.0022
$(i/6, 0)$	$0.9992 + 0.0164i$	$1.0006 + 0.0252i$	1.0015
$(i/7, 0)$	$0.9994 + 0.0140i$	$1.0004 + 0.0216i$	1.0012
$(i/8, 0)$	$0.9995 + 0.0122i$	$1.0003 + 0.0189i$	1.0009
$(i/9, 0)$	$0.9996 + 0.0108i$	$1.0003 + 0.0168i$	1.0008
$(i/10, 0)$	$0.9997 + 0.0097i$	$1.0002 + 0.0151i$	1.0006

TABLE 18. Complex moment with a zero parameter for 14a1 with $D = 3 \cdot 10^6$.

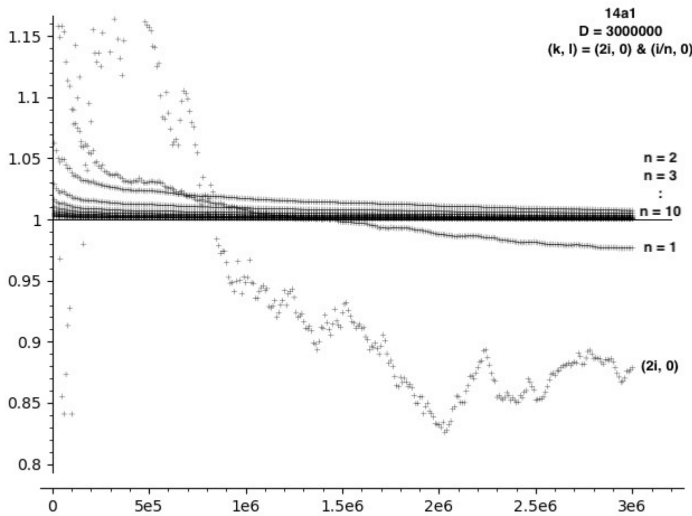

FIGURE 18. Complex moment with a zero parameter for 14a1 with $D \leq 3 \cdot 10^6$.

(k, ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(i, -i)$	3.2540	1.2527	60.801	48.511	22.948	0.4730
$(i/2, -i/2)$	1.3008	1.1201	3.1593	2.9982	2.6168	0.8728
$(i/3, -i/3)$	1.1204	1.0589	1.6918	1.6537	1.5707	0.9498
$(i/4, -i/4)$	1.0654	1.0342	1.3484	1.3314	1.2962	0.9736
$(i/5, -i/5)$	1.0411	1.0222	1.2120	1.2022	1.1827	0.9838
$(i/6, -i/6)$	1.0283	1.0156	1.1432	1.1369	1.1243	0.9889
$(i/7, -i/7)$	1.0207	1.0115	1.1257	1.0990	1.0902	0.9920
$(i/8, -i/8)$	1.0158	1.0088	1.0784	1.0750	1.0685	0.9940
$(i/9, -i/9)$	1.0124	1.0070	1.0615	1.0589	1.0539	0.9952
$(i/10, -i/10)$	1.0100	1.0057	1.0495	1.0474	1.0435	0.9962

TABLE 19. Complex moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 11a1 with $D = 3 \cdot 10^6$.

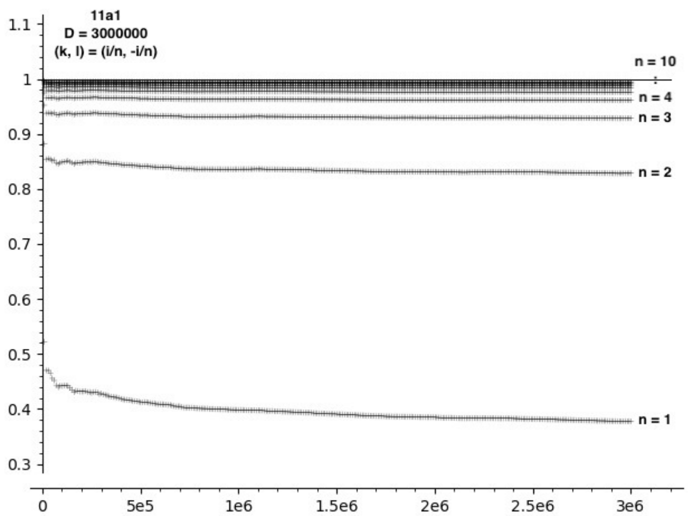

FIGURE 19. Complex moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 11a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell} + \log^{k\ell-1}$	moment	quotient
$(i,-i)$	3.2540	0.8050	39.070	32.025	18.186	0.5679
$(i/2,-i/2)$	1.3008	0.9587	2.7038	2.5815	2.3578	0.9133
$(i/3,-i/3)$	1.1204	0.9825	1.5697	1.5381	1.4877	0.9673
$(i/4,-i/4)$	1.0654	0.9903	1.2911	1.2765	1.2548	0.9830
$(i/5,-i/5)$	1.0411	0.9939	1.1783	1.1698	1.1576	0.9896
$(i/6,-i/6)$	1.0283	0.9957	1.1209	1.1153	1.1074	0.9929
$(i/7,-i/7)$	1.0207	0.9969	1.0876	1.0836	1.0781	0.9949
$(i/8,-i/8)$	1.0158	0.9976	1.0664	1.0634	1.0593	0.9962
$(i/9,-i/9)$	1.0124	0.9981	1.0522	1.0498	1.0466	0.9970
$(i/10,-i/10)$	1.0100	0.9985	1.0421	1.0402	1.0377	0.9976

TABLE 20. Complex moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 14a1 with $D = 3 \cdot 10^6$.

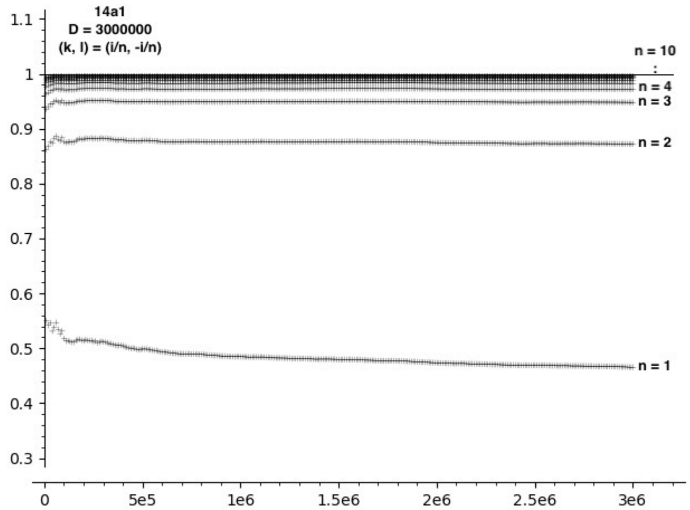

FIGURE 20. Complex moments of $\frac{L(1/2,E,\chi)}{L(1/2,E,\overline{\chi})}$ for 14a1 with $D \leq 3 \cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**11a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
(i,i)	$\Gamma(0)(0.1038+0.8024i)$	$-0.0960+0.5556i$	$-0.0306+0.0013i$	$-0.0379+0.00202i$	$-0.04055+0.00703i$	1.0867
$(i/2,i/2)$	$0.9343+0.2785i$	$0.8005+0.2624i$	$0.2803+0.1944i$	$0.2962+0.0232i$	$0.2984+0.2000i$	1.0000
$(i/3,i/3)$	$0.9374+0.09533i$	$0.9172+0.1672i$	$0.5799+0.1677i$	$0.5934+0.1707i$	$0.5970+0.1673i$	1.0040
$(i/4,i/4)$	$0.9550+0.04395i$	$0.9546+0.1232i$	$0.7358+0.1296i$	$0.7453+0.1307i$	$0.7410+0.1301i$	0.9943
$(i/5,i/5)$	$0.9676+0.02366i$	$0.9713+0.0977i$	$0.8212+0.1029i$	$0.8279+0.1035i$	$0.8323+0.1046i$	1.0054
$(i/6,i/6)$	$0.9760+0.01410i$	$0.9801+0.0810i$	$0.8718+0.0848i$	$0.8767+0.0851i$	$0.8805+0.0872i$	1.0045
$(i/7,i/7)$	$0.9817+0.00905i$	$0.9855+0.0692i$	$0.9039+0.0719i$	$0.9076+0.0721i$	$0.9108+0.0746i$	1.0036
$(i/8,i/8)$	$0.9856+0.00614i$	$0.9889+0.0605i$	$0.9255+0.0634i$	$0.9284+0.0625i$	$0.9310+0.0652i$	1.0030
$(i/9,i/9)$	$0.9884+0.00435i$	$0.9912+0.0537i$	$0.9406+0.0551i$	$0.9429+0.0552i$	$0.9451+0.0579i$	1.0025
$(i/10,i/10)$	$0.9905+0.00319i$	$0.9929+0.0483i$	$0.9516+0.0493i$	$0.9535+0.0494i$	$0.9553+0.0520i$	1.0020

TABLE 21. Complex moments of $|L(1/2,E,\chi)|$ for 11a1 with $D=3\cdot 10^6$.

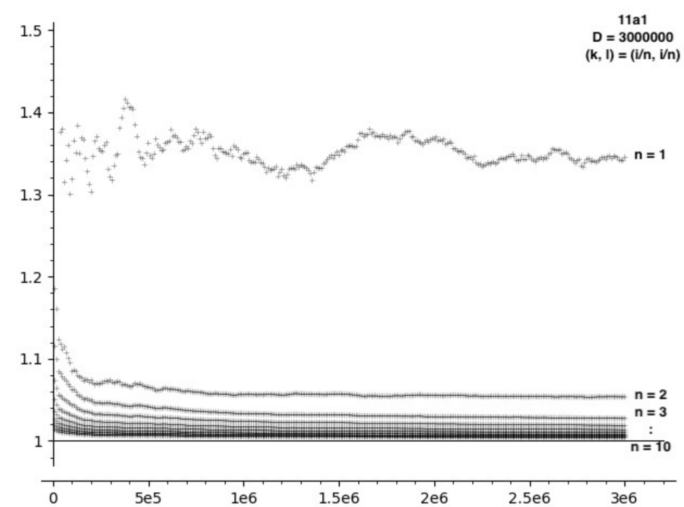

FIGURE 21. Complex moments of $|L(1/2,E,\chi)|$ for 11a1 with $D\le 3\cdot 10^6$.

(k,ℓ)	$g_{k,\ell}$	**14a1**				
		$a_{k\ell}$	$\log^{k\ell}$	$\log^{k\ell}+\log^{k\ell-1}$	moment	quotient
(i,i)	$\Gamma(0)(0.1038+0.8024i)$	$0.7853+0.5070i$	$-0.0218+0.0458i$	$-0.0243+0.0564i$	$-0.0188+0.0621i$	1.0561
$(i/2,i/2)$	$0.9343+0.2785i$	$0.9977+0.1264i$	$0.3725+0.1644i$	$0.3909+0.1702i$	$0.4000+0.1641i$	1.0142
$(i/3,i/3)$	$0.9374+0.09533i$	$1.0020+0.0728i$	$0.6407+0.1125i$	$0.6540+0.1138i$	$0.6689+0.1157i$	1.0226
$(i/4,i/4)$	$0.9550+0.04395i$	$1.0017+0.0518i$	$0.7749+0.0759$	$0.7838+0.0762i$	$0.7978+0.0842i$	1.0188
$(i/5,i/5)$	$0.9676+0.02366i$	$1.0012+0.0404i$	$0.8478+0.0550i$	$0.8540+0.0551i$	$0.8654+0.0654i$	0.9547
$(i/6,i/6)$	$0.9760+0.01410i$	$1.0009+0.0332i$	$0.8909+0.0424i$	$0.8954+0.0425i$	$0.9045+0.0534i$	1.0108
$(i/7,i/7)$	$0.9817+0.00905i$	$1.0007+0.0282i$	$0.9183+0.0344i$	$0.9216+0.0344i$	$0.9289+0.0452i$	1.0084
$(i/8,i/8)$	$0.9856+0.00614i$	$1.0005+0.0246i$	$0.9366+0.0288i$	$0.9392+0.0288i$	$0.9451+0.0391i$	1.0067
$(i/9,i/9)$	$0.9884+0.00435i$	$1.0004+0.0217i$	$0.9494+0.0248i$	$0.9516+0.0248i$	$0.9564+0.0345i$	1.0053
$(i/10,i/10)$	$0.9905+0.00319i$	$1.0003+0.0195i$	$0.9588+0.0218i$	$0.9605+0.0218i$	$0.9645+0.0309i$	1.0044

TABLE 22. Complex moments of $|L(1/2,E,\chi)|$ for 14a1 with $D=3\cdot 10^6$.

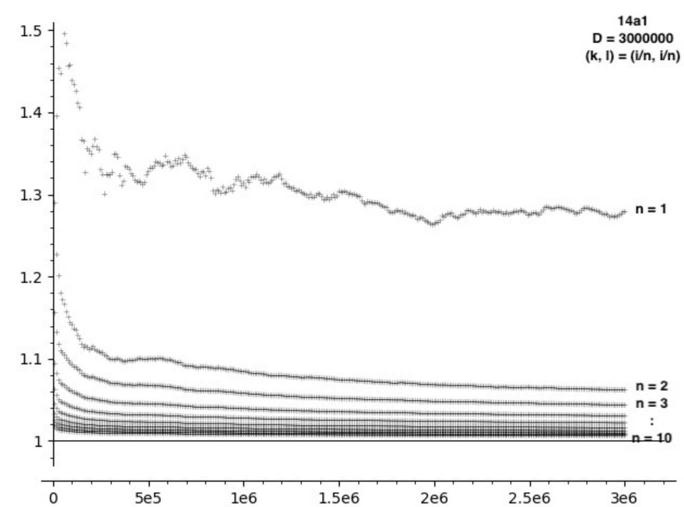

FIGURE 22. Complex moments of $|L(1/2,E,\chi)|$ for 14a1 with $D\le 3\cdot 10^6$.

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